ON THE PROXIMATE ORDER OF ENTIRE FUNCTIONS

R. S. L. Srivastava — O. P. Juneja

In this paper proximate orders of an entire function in terms of the maximum term of its Taylor series and the maximum modulus have been found.

1. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order ϱ and lower order λ; $M(r)$ denote the maximum modulus and $\mu(r)$ the maximum term of rank $\nu(r)$ in the Taylor expansion of $f(z)$ for $|z| = r$. Then $M(r)$, $\mu(r)$ and $\nu(r)$ are all positive and non-decreasing functions of r and

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log \mu(r)}{\log r}$$

$$= \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \nu(r)}{\log r} = \begin{matrix} \varrho \\ \lambda \end{matrix} .$$

When $\varrho = \lambda$, $f(z)$ is said to be of regular growth.

It is possible to find ([1], p. 64) a positive continons function $\varrho(r)$ having the following properties:

(i) $\varrho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\varrho'(r-0)$ and $\varrho'(r+0)$ exist.

(ii) $$\limsup_{r\to\infty} \varrho(r) = \varrho ;$$

(iii) $$\lim_{r\to\infty} r\,\varrho'(r) \log r = 0 ; \qquad \text{and}$$

(iv) $$\limsup_{r\to\infty} \frac{\log M(r)}{r^{\varrho(r)}} = 1.$$

Such a function $\rho(r)$ is called LINDELOF's Proximate Order for the entire function $f(z)$. In this paper we find proximate orders of $f(z)$ in terms of the maximum term $\mu(r)$ and the maximum modulus $M(r)$.

2. Theorem 1: *If $f(z)$ is an entire function of order ϱ, $(0 < \varrho < \infty)$ and $\nu(r)$ the rank of its maximum term $\mu(r)$ such that $\nu(r) \sim \Phi(r)\, r^{\varrho}$, where $\Phi(r)$ is a positive continuous function in (r_0, ∞) and $\Phi(cr) \sim \Phi(r)$ as $r \to \infty$ for every constant $c > 0$ then*

(I) *$f(z)$ is of regular growth;*

(II) $\lim\limits_{r\to\infty} \dfrac{\nu(r)}{\log \mu(r)} = \varrho$; *and*

(III) $\dfrac{\log \log \mu(r)}{\log r}$ *is a proximate order of $f(z)$.*

Proof: (I) Since $f(z)$ is of order ϱ, we have

$$\lim_{r\to\infty} \sup \frac{\log \nu(r)}{\log r} = \varrho .$$

Hence, for any $\varepsilon > 0$, we can find an $r_0 = r_0(\varepsilon)$ such that for every $r > r_0(\varepsilon)$

$$\log \nu(r) < (\varrho + \varepsilon) \log r$$

or,

$$\nu(r) < r^{\varrho+\varepsilon} . \tag{2.1}$$

Also, since

$$\nu(r) \sim \Phi(r)\, r^{\varrho}$$

we have, for any $\varepsilon > 0$,

$$(1-\varepsilon)\, \Phi(r)\, r^{\varrho} < \nu(r) < (1+\varepsilon)\, \Phi(r)\, r^{\varrho} \tag{2.2}$$

for $r > r_0'$.

Hence for sufficiently large r, we have from (2.1) and (2.2)

$$(1-\varepsilon)\, \Phi(r)\, r^{\varrho} < r^{\varrho+\varepsilon}$$

or

$$(1-\varepsilon)\, \Phi(r) < r^{\varepsilon}.$$

Taking logarithms and proceeding to limits we get, since $\Phi(r)$ is positive,

$$\lim_{r\to\infty} \frac{\log \Phi(r)}{\log r} = 0.$$

The condition $\nu(r) \sim \Phi(r)\, r^{\varrho}$ then gives

$$\lim_{r \to \infty} \frac{\log \nu(r)}{\log r} = \varrho .$$

So $f(z)$ is of regular growth.

(ii) To prove the second part, we have

$$\nu(r) \sim \Phi(r)\, r^{\varrho}$$

and so,

$$(\Phi(r) - \varepsilon)\, r^{\varrho} < \nu(r) < (\Phi(r) + \varepsilon)\, r^{\varrho} \qquad \text{for} \qquad r > r_0(\varepsilon).$$

Or,

$$\int_{r_0}^{r} (\Phi(r) - \varepsilon)\, r^{\varrho-1}\, dr < \int_{r_0}^{r} \frac{\nu(r)}{r}\, dr < \int_{r_0}^{r} (\Phi(r) + \varepsilon)\, r^{\varrho-1}\, dr \tag{2.3}$$

or,

$$\int_{r_0}^{r} \Phi(r)\, r^{\varrho-1}\, dr - \varepsilon \int_{r_0}^{r} r^{\varrho-1}\, dr < \int_{r_0}^{r} \frac{\nu(r)}{r}\, dr < \int_{r_0}^{r} \Phi(r)\, r^{\varrho-1}\, dr + \varepsilon \int_{r_0}^{r} r^{\varrho-1}\, dr .$$

Now, by Lemma V [2], the condition $\Phi(cr) \sim \Phi(r)$ involves

$$\int_{r_0}^{r} u^{\delta-1}\, \Phi(u)\, du \sim \frac{r^{\delta}}{\delta}\, \Phi(r)$$

for every positive δ, and so we get,

$$\frac{r^{\varrho}}{\varrho}\, \Phi(r) - \varepsilon \frac{r^{\varrho}}{\varrho} + O(1) < \int_{r_0}^{r} \frac{\nu(r)}{r}\, dr < \frac{r^{\varrho}}{\varrho}\, \Phi(r) + \varepsilon \frac{r^{\varrho}}{\varrho} - O(1). \tag{2.4}$$

Now it is known ([1], p. 31) that

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x}\, dx \tag{2.5}$$

Hence, (2.4) becomes

$$\frac{r^{\varrho}}{\varrho}\, \Phi(\varrho) - \varepsilon \frac{r^{\varrho}}{\varrho} + O(1) < \log \mu(r) - \log \mu(r_0) < \frac{r^{\varrho}}{\varrho}\, \Phi(r) + \varepsilon \frac{r^{\varrho}}{\varrho} - O(1).$$

Dividing by $\nu(r)$ and proceeding to limits, we have, since $\nu(r) \sim \Phi(r)\, r^{\varrho}$, (2.6) $$\lim_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} = \varrho.$$

(III) To prove (III), let

$$\varrho(r) = \frac{\log \log \mu(r)}{\log r}.$$

Then, $\lim_{r\to\infty} \varrho(r) = \varrho$, since $f(z)$ is of regular growth and order ϱ. Further, if

$$f(z) = \sum_{0}^{\infty} a_n z^n,$$

the maximum term $\mu(r)$ for $|z| = r$ is given by $\mu(r) = |a_{\nu(r)}| r^{\nu(r)}$. As $a_{\nu(r)}$, $\nu(r)$ are constants in intervals, have an enumerable number of discontinuities and change values at these discontinuities only, they are differentiable everywhere except at a set of measure zero and their derivatives vanish almost everywhere. Consequently, $\mu(r)$ and hence $\varrho(r)$ are also differentiable almost everywhere. Thus,

(2.7) $$r \varrho'(r) \log r = \frac{r \mu'(r)}{\mu(r) \log \mu(r)} - \frac{\log \log \mu(r)}{\log r}$$

where $\mu'(r)$ denotes the derivative of $\mu(r)$.

Since $\mu(r) = |a_{\nu(r)}| r^{\nu(r)}$, we get on differentiation

$$\frac{\mu'(r)}{\mu(r)} = \frac{\nu(r)}{r}$$

almost everywhere. Substituting in (2.7), we get,

$$r \varrho'(r) \log r = \frac{\nu(r)}{\log \mu(r)} - \frac{\log \log \mu(r)}{\log r} \to 0 \quad \text{as} \quad r \to \infty,$$

since by (2.6)

$$\lim_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} = \varrho = \lim_{r\to\infty} \frac{\log \log \mu(r)}{\log r}.$$

Also, from the definition of $\varrho(r)$

$$\frac{\log \mu(r)}{r^{\varrho(r)}} = 1$$

and since $\log \mu(r) \sim \log M(r)$ for functions of finite order, we have,

$$\lim_{r\to\infty} \frac{\log M(r)}{r^{\varrho(r)}} = \lim_{r\to\infty} \frac{\log \mu(r)}{r^{\varrho(r)}} = 1.$$

Thus, all the conditions for $\varrho(r)$ to be a proximate order are satisfied and so the theorem is proved.

3. If $W(r)$ be an indefinitely increasing function continuous in adjacent intervals, we know ([1], p. 27) that

$$\log M(r) = \log M(r_0) + \int_{r_0}^{r} x^{-1} W(x)\, dx.$$

Using this relation and proceeding as in Theorem 1, we can similarly prove the following:

Theorem 2: *If $f(z)$ is an entire function of order $\varrho\,(0 < \varrho < \infty)$, and $W(r)$ an indefinitely increasing positive function sach that $W(r) \sim \Phi(r)\, r^{\varrho}$ where $\Phi(r)$ is a positive continuous function in (r_0, ∞) and $\Phi(cr) \sim \Phi(r)$ for every constant $c > 0$, then*

(i) $\lim\limits_{r \to \infty} \dfrac{W(r)}{\log M(r)} = \varrho,$ *and*

(ii) $\dfrac{\log \log M(r)}{\log r}$ *is a proximate order of $f(z)$.*

REFERENCES

[1] G. Valiron : **Lectures on the General Theory of Integral Functions**, Chelsea, (1949).

[2] G. H. Hardy and W. W. Rogosinski : *Notes on Fourier Series (III)*, Quart. Journ. of Maths., **16**, (1945).

Department of Mathematics
Indian Institute of Technology
Kanpur — India

(Manuscript received April 25, 1963)

ÖZET

Bu makalede tam fonksiyonların «proximate» (komşu) mertebeteri, bahis konusu fonksiyonların Taylor serilerinin maksimum terimi ile maksimum modülleri cinsinden ifade edilmiştir.