ON THE FLOW OF TWO CONDUCTING INCOMPRESSIBLE IMMISCIBLE FLUIDS BETWEEN TWO NON - CONDUCTING PLATES

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The steady flow of two conducting incompressible and immiscible fluids between two non-conducting parallel plates has been considered in the presence of a transverse magnetic field. It has been shown that as the Hartmann numbers for the two fluids increase, the flux, the inter-faee velocity and the skin frictions at the two plates decrease. It has been shown that unlike the case ot non-conducting fluids where the velocity maximum occurs in the less viscous fluid, here the velocity - maximum can occur in the more viscous fluid also.

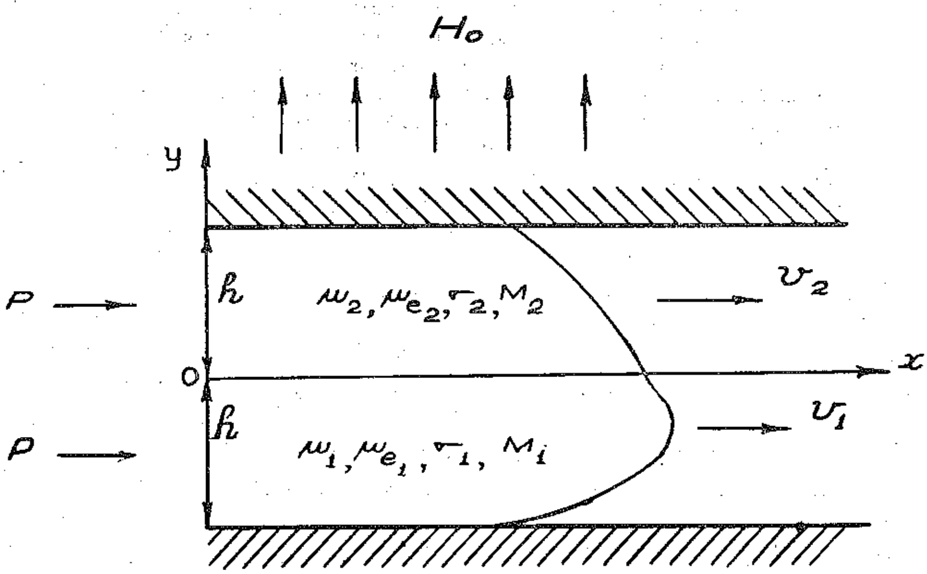
1. Introduction

The velocity profile due to the flow of two incompressible immiscible fluids between two parallel plates was obtained in [1] and it was shown there that when the heights of the two fluids are equal, the velocity - maximum occurs in the fluid of smaller viscosity. The discussion was generalised by the present authors [2] to the case of the flow of n-incompressible immiscible fluids occupying different heights between two parallel plates and it was shown that a unique velocity maximum always exists.

In the present paper we have considered the flow of two conducting incompressible and immiscible fluids occupying equal heights between two parallel non-conducting plates in the presence of a transverse magnetic field. It has been shown that as the Hartmann numbers for the two fluids increase, the flux, the inter-face velocity and the skin-frictions at the two plates decrease. These results are consistent with the phenomenon [3] that a magnetic field gives a certain degree of «rigidity» to a fluid. We also obtain another interesting result which is also consistent with this phenomenon. We prove that while for non-conducting fluids, the velocity maximum always occurs in the less viscous fluid, in the case of conducting fluids it can occur in the more viscous fluid also, if its conductivity is less than that of the less viscous fluids. In fact we fiud that for given Hartmann numbers there is a critical value of the ratio of the two viscosity coefficients for which the velocity maximum will occur at the interface and the velocity maximum will occur iu one fluid or the other depending on whether the actual ratio of the viscosity coefficients is less than or greater than this critical value.

2. Determination of the velocity profile

Let $2h$ be the distance between the plates and let μ_1, μ_2, σ_1, σ_2, μ_{e_1}, μ_{e_2} denote the coefficients of viscosity, conductivity and permeability respectively of the two fluids each occupying a height h. We take the x-axis along the interface and the y-axis perpendicular to it drawn into the second fluid. Also let a constant pressure gradient ϱ be applied to both the fluids.



The equations governing the hydromagnetic flow are [4]

(1)
$$\frac{d^2 v_i}{dy^2} - \frac{M_i^2}{h^2} v_i = -\frac{P}{\mu_i}, \qquad [i = 1, 2]$$

where

(2)
$$M_i = H_0 \, h \, \mu_{e_i} \sqrt{\frac{\sigma_i}{\mu_i}} \qquad [i = 1, 2]$$

are the Hartmann numbers and v_1, v_2 denote the velocities of the two fluids. The boundary conditions are

(3)
$$\begin{aligned} v_1 &= 0 \quad \text{when} \quad y = -h, \\ v_1 &= u_0 \quad \text{when} \quad y = 0, \\ v_2 &= 0 \quad \text{when} \quad y = +h, \\ v_2 &= u_0 \quad \text{when} \quad y = 0, \end{aligned}$$

where u_0 is the common interface velocity (to be determined). Solving (1), subject to (3), we get

(4)
$$\frac{v_1 \mu_1}{P h^2} = \frac{1}{M_1^2} \left\{ 1 - \frac{\sinh M_1\left(1+\frac{y}{h}\right) - \sinh \frac{M_1 y}{h}}{\sinh M_1} \right\} + \frac{u_0 \mu_1}{P h^2} \frac{\sinh M_1\left(1+\frac{y}{h}\right)}{\sinh M_1} \qquad [-h \leq y \leq 0],$$

and

(5)
$$\frac{v_2 \mu_2}{P h^2} = \frac{1}{M_2^2} \left\{ 1 - \frac{\sinh M_2\left(1-\frac{y}{h}\right) + \sinh M_2 \frac{y}{h}}{\sinh M_2} \right\} + \frac{u_0 \mu_2}{P h^2} \frac{\sinh M_2\left(1-\frac{y}{h}\right)}{\sinh M_2} \qquad [0 \leq y \leq h].$$

3. Determination of interface velocity, flux and skin-frictions at the two plates

To determine u_0 in the ordinary hydrodynamic case [1] we use the continuity of the shear stress at the interface. In the present case we can include the contribution of the magnetic field to the shear stress, but since the induced magnetic field is continous at the interface, this factor does not change the boundary condition which gives

(6)
$$\left(\mu_1 \frac{dv_1}{dy}\right)_{y=0} = \left(\mu_2 \frac{dv_2}{dy}\right)_{y=0}$$

and then by (4) and (5), we get

(7)
$$\frac{u_0 \mu_1}{P h^2} = \frac{M_2 \tanh \frac{M_1}{2} + M_1 \tanh \frac{M_2}{2}}{M_1 M_2 (M_1 \coth M_1 + \lambda M_2 \coth M_2)}$$

where

(8)
$$\lambda = \frac{\mu_2}{\mu_1}.$$

The total flux Q is given by

(9)
$$Q = \int_{-h}^{0} v_1\, dy + \int_{0}^{h} v_2\, dy = Q_1 + Q_2 \qquad \text{(say)}$$

where from (4) and (5),

(10) $$\frac{Q_1 \mu_1}{Ph^3} = \frac{2}{M_1^3}\left[\frac{1}{2} M_1 - \tanh \frac{1}{2} M_1\right] + \frac{u_0 \mu_1}{2Ph^2} \frac{\tanh \frac{1}{2} M_1}{\frac{1}{2} M_1},$$

(11) $$\frac{Q_2 \mu_2}{Ph^3} = \frac{2}{M_2^3}\left[\frac{1}{2} M_2 - \tanh \frac{1}{2} M_2\right] + \frac{u_0 \mu_2}{2Ph^2} \frac{\tanh \frac{1}{2} M_2}{\frac{1}{2} M_2}.$$

The skin-frictions at the two plates are given by:

(12) $$P_1 = \left(\mu_1 \frac{dv_1}{dy}\right)_{y=-h} = Ph \left\{ \frac{1}{2} \frac{\tanh \frac{1}{2} M_1}{\frac{1}{2} M_1} + \frac{u_0 \mu_1}{Ph^2} \frac{M_1}{\sinh M_1} \right\},$$

(13) $$P_2 = \left(\mu_2 \frac{dv_2}{dy}\right)_{y=h} = -Ph \left\{ \frac{1}{2} \frac{\tanh \frac{1}{2} M_2}{\frac{1}{2} M_2} + \frac{u_0 \mu_2}{Ph^2} \frac{M_2}{\sinh M_2} \right\}.$$

4. Results about interface velocity, flux and skin frictions

(*i*) Equation (7) can be written as

(14) $$\frac{2u_0}{Ph^2} = \frac{\dfrac{\tanh \frac{1}{2} M_1}{\frac{1}{2} M_1} + \dfrac{\tanh \frac{1}{2} M_2}{\frac{1}{2} M_2}}{\mu_1 \dfrac{M_1}{\tanh M_1} + \mu_2 \dfrac{M_2}{\tanh M_2}}.$$

It is obvious that the right hand side of (14) does not change if M_1, μ_1 are interchanged with M_2, μ_2. This is a result which we expect. We can see similarly from the pairs of equations (4) and (5), (10) and (11), (12) and (13) that v_1, Q_1 and P_1 are interchanged with v_2, Q_2 and P_2 if M_1, μ_1 and y are interchanged with M_2, μ_2 and $-y$ respectively.

(*ii*) Since

(15) $$\left\{ \begin{array}{ll} \dfrac{d}{dM}\left[\dfrac{\tanh M}{M}\right] = \dfrac{2M - \sinh 2M}{2M^2 \cosh^2 M}; & \dfrac{d}{dM}[2M - \sinh 2M], \\ {[2M - \sinh 2M]}_{M=0} = 0, & = 2\,[1 - \cosh 2M] \\ & \leq 0, \end{array} \right.$$

it follows that $(\tanh M)/M$ is a decreasing function of M when $M > 0$.

Therefore we get from (14) that the interface velocity decreases as M_1, M_2

increase. In particular it will decrease if the applied magnetic field increases or we use fluids with greater electrical conductivities. To see the effect of an increase of μ_1, μ_2 we write (14) as

$$\frac{2u_0}{Ph^2} = \frac{\dfrac{2\sqrt{\mu_1}}{K}\tanh\dfrac{1}{2}\dfrac{K}{\sqrt{\mu_1}} + \dfrac{\tanh\frac{1}{2}M_2}{\frac{1}{2}M_2}}{K\sqrt{\mu_1}\coth\dfrac{K}{\sqrt{\mu_1}} + \mu_2 M_2 \coth M_2}. \tag{14'}$$

We keep K, M_2, μ_2 fixed and increase μ_1 then it can he shown that both

$$\sqrt{\mu_1}\tanh\frac{1}{2}\frac{K}{\sqrt{\mu_1}} \qquad \text{and} \qquad \sqrt{\mu_1}\coth\frac{K}{\sqrt{\mu_1}}$$

increase as μ_1 increases, hut the first term increases at a slower rate as its one factor increases while the other decreases, while in the second term both factors increase. Thus the interface velocity decreases as the viscosity coefficients increase.

(*iii*) Since

$$\left\{\begin{aligned} &\frac{d}{dM}\left[\frac{M-\tanh M}{M^3}\right] = \frac{3\tanh M - 2M - M\operatorname{sech}^2 M}{M^4}, \\ &\qquad\qquad [3\tanh M - 2M - M\operatorname{sech}^2 M]_{M=0} = 0, \\ &\frac{d}{dM}[3\tanh M - 2M - M\operatorname{sech}^2 M] \\ &\qquad\qquad = -[2\tanh^2 M - 2\tanh M\operatorname{sech}^2 M] \leq 0 \end{aligned}\right. \tag{16}$$

it follows that

$$\frac{M-\tanh M}{M^3}$$

is a decreasing function of M, $M > 0$. Since

$$\frac{\tanh M}{M}$$

is also a decreasing function of M we find, on using the result (*ii*) that both Q_1, Q_2 decrease as M_1, M_2 increase.

(*iv*) Since

$$(17)\quad \begin{cases} \dfrac{d}{dM}\left[\dfrac{M}{\sinh M}\right] = \dfrac{\sinh M - M\cosh M}{\sinh^2 M} \\ \qquad [\sinh M - M\cosh M]_{M=0} = 0 \\ \dfrac{d}{dM}[\sinh M - M\cosh M] \\ \qquad = -M\sinh M \leq 0. \end{cases}$$

It follows that

$$\frac{M}{\sinh M}$$

is also a decreasing function of M. From (12) and (13), we get that both P_1 and P_2 decrease as M_1, M_2 increase.

(v) If we take the limits as M_1, M_2 tend to zero of (4), (5), (7), (10), (11), (12) and (13) we get the same results as those of [1].

5. The velocity maximum

To obtain the velocity maximum we have to find where the derivative of the velocity vanishes. For this point we get from (4) and (5),

$$(18)\quad \frac{y_1}{h} = \frac{1}{2M_1}\log \frac{1 - e^{-M_1}\left[1 - \dfrac{M_1}{M_2}\,\dfrac{M_2\tanh\dfrac{M_1}{2} + M_1\tanh\dfrac{M_2}{2}}{M_1\coth M_1 + \lambda M_2\coth M_2}\right]}{1 - e^{M_1}\left[1 - \dfrac{M_1}{M_2}\,\dfrac{M_2\tanh\dfrac{M_1}{2} + M_1\tanh\dfrac{M_2}{2}}{M_1\coth M_1 + \lambda M_2\coth M_2}\right]}$$

$$(19)\quad \frac{y_1}{h} = \frac{1}{2M_1}\log\frac{1 - e^{-M_1}\Phi_1(M_1, M_2, \lambda)}{1 - e^{M_1}\Phi_1(M_1, M_2, \lambda)} = \frac{1}{2M_1}\log\frac{F_{11}}{F_{12}} \qquad \text{(say)},$$

$$(20)\quad \frac{y_2}{h} = \frac{1}{2M_2}\log \frac{1 - e^{M_2}\left[1 - \dfrac{M_2\lambda}{M_1}\,\dfrac{M_2\tanh\dfrac{M_1}{2} + M_1\tanh\dfrac{M_2}{2}}{M_1\coth M_1 + \lambda M_2\coth M_2}\right]}{1 - e^{-M_2}\left[1 - \dfrac{M_2\lambda}{M_1}\,\dfrac{M_1\tanh\dfrac{M_1}{2} + M_1\tanh\dfrac{M_2}{2}}{M_1\coth M_1 + \lambda M_2\coth M_2}\right]}$$

$$(21)\quad \frac{y_2}{h} = \frac{1}{2M_2}\log\frac{1 - e^{M_2}\Phi_2(M_1, M_2, \lambda)}{1 - e^{-M_2}\Phi_2(M_1, M_2, \lambda)} = \frac{1}{2M_2}\log\frac{F_{21}}{F_{22}} \qquad \text{(say)}.$$

If

$$-1 < \frac{y_1}{h} < 0,$$

the velocity maximum will be in the first layer, while if

$$0 < \frac{y_2}{h} < 1,$$

it will be in the second layer.

The velocity maximum will be in the common interface if $\lambda = \lambda^*$, where

(22) $$\lambda^* = \frac{f(M_2)}{f(M_1)}$$

and

(23) $$f(M) = \frac{\cosh M - 1}{M^2 \cosh M}.$$

We observe the following facts:

(*i*) If λ increases, $\Phi_1(M_1, M_2, \lambda)$ increases so that both F_{11}, F_{12} decrease but F_{12} decreases more so that F_{11}/F_{12} increases. If

$$\lambda > \lambda^*, \quad \frac{F_{11}}{F_{12}} > 1,$$

y_1/h becomes positive and maximum velocity cannot occur in the first fluid.

(*ii*) If λ increases, $\Phi_2(M_1, M_2, \lambda)$ decreases so that both F_{21} and F_{22} increase, but F_{21} increases more, so that F_{21}/F_{22} increases. If

$$\lambda > \lambda^*, \quad \frac{F_{21}}{F_{22}} > 1$$

so that y_2/h becomes positive and a velocity maximum can occur in the second fluid.

(*iii*) If

$$\lambda < \lambda^*, \quad \frac{F_{11}}{F_{12}} < 1,$$

y_1/h is negative and velocity maximum can occur in the first fluid.

(*iv*) If

$$\lambda < \lambda^*, \quad \frac{F_{21}}{F_{22}} < 1,$$

y_2/h is negative and a velocity maximum cannot occur in the second fluid.

(*v*) The velocity vanishes at both the plates, and both the velocities and its derivative are continuous at all points except at the interface where the derivative can change in magnitude hut not in sign. Remembering all the above faets, we conclude that there is a unique velocity maximum and it occurs in the first fluid if $\lambda > \lambda^*$, in the second fluid if $\lambda > \lambda^*$ and at the common interface if $\lambda = \lambda^*$. As λ increases steadily from 0 to ∞, the velocity maximum goes from a point in the first fluid given by $y = y_{10}$ to a point in the second fluid given by $y = y_{20}$ where,

$$\frac{y_{10}}{h} = -1 + \frac{1}{2M_1} \log \frac{e^{M_1} - 1 + \dfrac{M_2 \tanh \frac{M_1}{2} + M_1 \tanh \frac{M_2}{2}}{M_2 \coth M_1}}{e^{-M_1} - 1 + \dfrac{M_2 \tanh \frac{M_1}{2} + M_1 \tanh \frac{M_2}{1}}{M_2 \coth M_1}}, \tag{24}$$

$$\frac{y_{20}}{h} = -1 - \frac{1}{2M_2} \log \frac{e^{M_2} - 1 + \dfrac{M_2 \tanh \frac{M_1}{2} + M_1 \tanh \frac{M_2}{2}}{M_1 \coth M_2}}{e^{-M_2} - 1 + \dfrac{M_2 \tanh \frac{M_1}{2} + M_1 \tanh \frac{M_2}{2}}{M_1 \coth M_2}}. \tag{25}$$

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ÖZET

Sıkışmayan ve karışmayan iki iletken sıvının iletken olmayan iki levha arasındaki düzgün akışı, akış istikametine dik bir manyetik alan mevcut olması halinde incelenmektedir. İki sıvının Hartmann sayıları büyüdükçe akış, ayırma yüzeyi hız ve levhalardaki yüzey sürtünmesi azaldığını gösterilmiştir. İletken olmayan sıvılarda hız maksimumu en az viskoziteyi haiz sıvıda vuku bulmasına karşı incelemekte olan hız maksimumu en yüksek viskoziteyi haiz sıvıda da vuku bulabilir.