ON UNIVALENT FUNCTIONS

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In this paper the problem of determining an upper bound of the absolute value of the difference between the moduli of two successive coefficients of a univalent function is considered in the special case of a so-called k-fold symmetric starlike univalent function. It is shown that the result found here is a refinement of another result for $k=1$ due to G. M. GOLUSIN.

Introduction. Let $\mathfrak{S}$ be the class of all functions $f(z)$, analytic and univalent in the circle $|z|<1$, and normalized as $f(0)=0$, $f'(0)=1$; and let $\mathfrak{S}^*$ be its subclass which consists of all functions starlike with respect to $z=0$. Further, let $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ be a function from the class $\mathfrak{S}^*$. G. M. GOLUSIN proved that [1])

$$(1) \qquad \big|\, |a_{n+1}| - |a_n| \,\big| < C \; ; n=1,2,3,\ldots,$$

where C denotes a positive absolute constant[2]) [3].

Now, let $k>1$ be a given integer, and let us consider the class $\mathfrak{S}_k$ of functions of the form $f_k(z)=z+a_2^{(k)}z^{k+1}+\cdots+a_n^{(k)}z^{(n-1)k+1}+\cdots$ as a subclass of $\mathfrak{S}$. One may surmise, as a refinement of (1), that in general

$$(2) \qquad \big|\, |a_{n+1}^{(k)}| - |a_n^{(k)}| \,\big| < C(k)\, n^{\frac{1}{k}-1} \quad ; n=1,2,3.\ldots,$$

where $C(k)$ denotes a positive constant dependent only on k[2]). In this paper we shall prove that (2) is valid at least for the subclass $\mathfrak{S}_k^* = \mathfrak{S}_k \cap \mathfrak{S}^*$.

Incidentally, at the same time we shall prove also that

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1) W. K. HAYMAN later proved that this inequality is valid even for the more general class of «areally univalent» functions. (See the Journal of the London Mathematical Society, April 1963 issue.)

2) Throughout the paper C will always denote a positive absolute constant: while $C(k)$, $C_1(k)$, $C_2(k)$ will denote positive constants dependent only on k. Neither of the foregoing entries will necessarily refer to the same constant from one occurence to another.

$$\frac{1}{n}\sum_{\lambda=1}^{n}\Big|\,|a_{\lambda+1}^{(k)}|-|a_{\lambda}^{(k)}|\,\Big|<C(k)\,n^{\frac{1}{k}-1}.$$

1. In the first place we shall quote certain classical results, already known, as shown in the following lemmas:

Lemma 1-1: For any $f(z)\in\mathfrak{S}$ and for $|z_1|=\varrho_1<1$, $|z_2|=\varrho_2<1$ the inequality

$$|z_1-z_2|\cdot\frac{\sqrt{1-\varrho_1^2}\cdot\sqrt{1-\varrho^2}}{\varrho_1\varrho_2}\leq\left|\frac{1}{f(z_1)}-\frac{1}{f(z_2)}\right|$$

holds [1].

Lemma 1-2: If $f(z)\in\mathfrak{S}^*$, then the coefficients of the Taylorian expansion, about the origin, of the function $z\dfrac{f'(z)}{f(z)}$ are bounded. More precisely, if $z\dfrac{f'(z)}{f(z)}=1+d_1z+d_2z^2+\cdots+d_nz^n+\cdots$, then $|d_n|\leq 2$ for all n. (For the proof see [5]; also may be obtained as a simple consequence of Hilfsatz 2, [4], p. 167, merely by considering necessary and sufficient condition for being starlike.)

Lemma 1-3: If $f_k(z)=z+a_2^{(k)}z^{k+1}+\cdots+a_n^{(k)}z^{(n+1)k+1}+\cdots$ is any function from the class $\mathfrak{S}_k^*$, the inequality

$$|a_n^{(k)}|<C(k)\,n^{\frac{2}{k}-1}$$

holds [2].

Lemma 1-4: If $f_k(z)=z+a_2^{(k)}z^{k+1}+\cdots+a_n^{(k)}z^{(n-1)k+1}+\cdots$ is any function from the class $\mathfrak{S}_k$, the inequality

$$\sum_{\lambda=1}^{n}|a_{\lambda}^{(k)}|<C(k)\,n^{\frac{2}{k}}$$

holds [2].

We shall mention also the following lemma which seems significant to demonstrate the inner relation between the class $\mathfrak{S}^*$ and its subclass $\mathfrak{S}_k^*$:

Lemma 1-5: If $f(z)\in\mathfrak{S}^*$ then $\sqrt[k]{f(z^k)}\in\mathfrak{S}_k^*$. Conversely, if $f_k(z)\in\mathfrak{S}_k^*$ then $[f_k(\sqrt[k]{z})]^k\in\mathfrak{S}$. (Certainly, for $\sqrt[k]{\ }$ a definite branch is choosen.) (For proof of the first part see [4], p. 40; the proof of the second part is similar.)

Lemma 1-6: For any $f(z)\in\mathfrak{S}$ we have

$$|f(re^{i\vartheta})|\leq\frac{r}{(1-r)^2};\ 0<r<1.$$

(The well-known Modulus Theorem [4], p. 44.)

Lemma 1-7: For any $f(z) \in \mathfrak{S}$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} | f(re^{i\vartheta}) |^2 \, d\vartheta \leqq \int_0^r \frac{M(\varrho)}{\varrho} \, d\varrho \, ; \; 0 < r < 1,$$

where $M(\varrho) = \underset{|z|=\varrho}{\text{Max}} \, | f(z) |$. (A special case of Hilfsatz 1, [4]), p. 151)

2. As the next step we shall obtain certain preliminary results which are straightforward consequences of foregoing lemmas:

In the first place, it is almost obvious from Lemma 1—5 that, for any given k, to each $f_k(z) \in \mathfrak{S}_k$ there corresponds an $f(z) \in \mathfrak{S}_k$ such that the identity

$$(2\text{—}1) \qquad [f_k(z)]^k \equiv f(z^k)$$

holds within the unit circle.

Next, we shall state and prove the following lemma:

Lemma 2-1: If $f(z) \in \mathfrak{S}^*$ then the corresponding $f_k(z) \in \mathfrak{S}_k^*$ and vice versa.

Proof: In fact, one can write from (2—1)

$$(2\text{—}2) \qquad z\frac{f_k'(z)}{f_k(z)} \equiv z^k \frac{f'(z^k)}{f(z^k)} \, .$$

A necessary and sufficient condition for $f(z) \in \mathfrak{S}^*$ is that, for every $\varrho \; (0 < \varrho < 1)$ and every z for which $| z | = \varrho$, $Re\left(z\frac{f'(z)}{f(z)} \right) > 0$. Further, it is obvious that $Re\left(z\frac{f'(z)}{f(z)} \right) > 0$, for above described ϱ' s and z' s, if and only if $Re\left(z^k\frac{f'(z^k)}{f(z^k)} \right) < 0$ for the same ϱ' s and z' s. On the other hand, it is evident from (2—2) that $Re\left(z^k\frac{f'(z^k)}{f(z^k)} \right) > 0$ if and only if $Re\left(z\frac{f_k'(z)}{f_k(z)} \right) > 0$. This implies that $Re\left(z\frac{f'(z)}{f(z)} \right) > 0$ if and only if $Re\left(z\frac{f_k'(z)}{f_k(z)} \right) > 0$ which means that $f(z) \in \mathfrak{S}^*$ if and only if $f_k(z) \in \mathfrak{S}_k^*$.

Now, let $f_k(z) \in \mathfrak{S}_k^*$. According to the foregoing lemma, the corresponding $f(z) \in \mathfrak{S}^*$. Let us set $R(\varrho, \vartheta) = | f(\varrho e^{i\vartheta}) |$ and $\Phi(\varrho, \vartheta) = arg \, f(\varrho e^{i\vartheta})$; and consider the CAUCHY-RIEMANN Identity $\frac{\partial R}{\partial \varrho} = \frac{R}{\varrho} \cdot \frac{\partial \Phi}{\partial \vartheta}$. Since $f(z)$ is starlike, for every fixed $\varrho \; (0 < \varrho < 1)$, $\frac{\partial \Phi}{\partial \vartheta} > 0$; and, as a result, for every fixed ϑ, $R(\varrho, \vartheta)$ is a monotonically increasing function of ϱ. Suppose that $| z_2 | < z_1 |$; then, since R is monotonically increasing, $| f(z_2) | < | f(z_1) |$. Hence, one can write from Lemma 1—1

(2—3) $$| (z_1 - z_2) f(z_2) | \leq \frac{2 \varrho_1 \varrho_2}{\sqrt{1-\varrho_1^2} \cdot \sqrt{1-\varrho_2^2}}$$

where $| z_2 | = \varrho_2 < \varrho_1 = | z_1 |$.

Next, let us try to make an estimate for $| (z_1^k - z_2^k) \cdot f_k(z) |$. For this purpose write after (2—1)

$$| (z_1^k - z_2^k) \cdot f_k(z) | \leq [\, | (z_1^k - z_2^k) \cdot f(z^k) | \,]^{\frac{1}{k}} (| z_1 |^k + | z_2 |^k)^{1 - \frac{1}{k}} .$$

Using (2—3) on the right-hand side of foregoing inequality, one can write, with z_2 replaced by z and ϱ_2 replaced by ϱ,

$$| (z^k - z_1^k) \cdot f_k(z) | < \frac{2 \varrho_1^k \varrho}{(- \varrho_1^{2k})^{\frac{1}{2k}} (1 - \varrho^{2k})^{\frac{1}{2k}}}$$

where $| z | = \varrho < \varrho_1 = | z_1 |$.

Finally let us mention and prove following two lemmas:

Lemma 2-2: For any $f(z) \in \mathfrak{S}^*$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| z \frac{f'(z)}{f(z)} \right|^2 d\vartheta \leq 1 + \frac{4 r^2}{1 - r^2} ; \; z = re^{i\vartheta}, 0 < r < 1.$$

Proof: In fact, let $z \dfrac{f'(z)}{f(z)} = 1 + d_1 z + \cdots + d_n z^n + \cdots$. Accordingly, in view of the Parseval Identity and the Lemma 1—2, one can write

$$\frac{1}{2\pi} \int_0^{2\pi} \left| z \frac{f'(\zeta)}{f(\zeta)} \right|^2 \leq 1 + 4 (r^2 + r^4 + \cdots + r^{2n} + \cdots)$$

from which the lemma follows immediately.

Lemma 2-3: For any $f_k(z) = z + a_1^{(k)} z^{k+1} + \cdots + a_n^{(k)} z^{(n-1)k+1} + \cdots$ from the class $\mathfrak{S}_k$ we have

$$\sum_{\lambda=1}^{n} | a_\lambda^{(k)} |^2 r^{2\lambda} < \begin{cases} \dfrac{C(k)}{(1-r)^{\frac{4}{k}-1}}, & \text{for } k = 1, 2, 3 ; \\[2ex] C \log \dfrac{1}{1-r}, & \text{for } k = 4 ; \\[2ex] C(k), & \text{for any } k < 4. \end{cases}$$

Proof: In fact, Lemma 1—6 and the identity (2—1) suggest that

$\underset{|z|=\delta}{\text{Max}} |f_k(z)| = \frac{\varrho}{(1-\varrho^k)^{\frac{2}{k}}}$. Thus, the lemma at once deduced as a simple consequence of PARSEVAL identity and the Lemma 1—7.

3. Now, let us associate with each

$$f_k(z) = z + a_2^{(k)} z^{k+1} + \cdots + a_n^{(k)} z^{(n-1)k+1} + \cdots$$

the function $a(\zeta) = 1 + a_2^{(k)} \zeta + \cdots + a_n^{(k)} \zeta^{n-1} + \cdots$ which is analytic in the circle $|\zeta| < 1$, and, which together with $f(z)$ satisfies the identity

$$f_k(z) \equiv z\, u(z^k). \tag{3—1}$$

Thus, using (3—1) and setting $z^k = \zeta$, $z_1^k = \zeta_1$, $\varrho_1^k = r_1$, $\varrho^k = r$, and

$$\varphi(\zeta) = (\zeta - \zeta_1)\, a(\zeta), \tag{3—2}$$

the inequality (2—4) may he replaced hy

$$|\varphi(\zeta)| < \frac{2r_1}{(1-r_1^2)^{\frac{1}{2k}} \cdot (r_1^2 - r^2)^{\frac{1}{2k}}} \tag{3—2}$$

where $|\zeta| = r < r_1 = |\zeta_1|$.

Now, let $\varphi(\zeta) = b_0 + b_1 \zeta + \cdots + b_n \zeta^n + \cdots$. Equating the coefficients of ζ^n on both sides of (3—2), we have

$$b_n = a_n^{(k)} - \zeta_1 a_{n+1}^{(k)}; \qquad n = 1, 2, 3, \cdots . \tag{3—3}$$

Finally let us mention for later use, the inequality,

$$\left| |a_{n+1}^{(k)}| - |a_n^{(k)}| \right| \leq \left| b_n | + | a_{n+1}^{(k)} \right| (1 - r_1) \tag{3—4}$$

which may be derived from (3—3) hy simple reasonings.

4. Next, let us define the function $g(\zeta) = \zeta \frac{\varphi'(\zeta)}{\varphi(\zeta)}$ which is analytic in the circle $|\zeta| < r_1$, and, which together with $\varphi(\zeta)$ satisfies the identity

$$\zeta \varphi'(\zeta) \equiv \varphi(\zeta) \cdot g(\zeta). \tag{4—1}$$

Accordingly, as a result of the combination of (2—1), (3—1), (3—2) and (4—1), one may have

$$g(\zeta) = \frac{1}{k}\left[\zeta \frac{f'(\zeta)}{f(\zeta)} - 1\right] + \frac{\zeta}{\zeta - \zeta_1}. \tag{4—2}$$

Then, applying the simple inequality $|a+b|^2 \leq 2[|a|^2 + |b|^2]$ to the right-hand side of (4—2), one can write

$$\frac{1}{2\pi}\int_0^{2\pi}|g(\zeta)|^2 d\vartheta \leq \frac{2}{k^2}\left[\int_0^{2\pi}\left|\zeta\frac{f'(\zeta)}{f(\zeta)}-1\right|^2 d\vartheta\right]+\frac{r^2}{\pi}\int_0^{2\pi}\frac{d\vartheta}{|\zeta-\zeta_1|^2}; \quad \zeta = re^{i\vartheta},$$

and, considering the simple integral $\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{|a-b\,e^{i\vartheta}|^2}=\frac{1}{|a|^2-|b|^2}$ together with the Lemma 2—2, one obtains

$$\text{(4—3)} \qquad \frac{1}{2\pi}\int_0^{2\pi}|g(\zeta)|^2 d\vartheta < \frac{C(k)\,r^2}{r_1^2-r^2}; \quad \zeta = re^{i\vartheta}.$$

On the other hand, one can write from (4—1)

$$\frac{1}{2\pi}\int_0^{2\pi}|\varphi'(\zeta)|^2 d\vartheta \leq \frac{1}{r^2}\max_{|\zeta|=r}|\varphi(\zeta)|^2\cdot\frac{1}{2\pi}\int_0^{2\pi}|g(\zeta)|^2 d\vartheta; \quad \zeta = re^{i\vartheta},$$

and considering (3—2) and (4—3)

$$\text{(4—4)} \qquad \frac{r}{2\pi}\int_0^{2\pi}|\varphi'(\zeta)|^2 d\vartheta < \frac{C(k)\,r}{(1-r_1^2)^{\frac{1}{k}}\cdot(1-r^2)^{\frac{1}{k}}}.$$

Next, applying to the left-hand side the PARSEVAL identity, and, integrating both sides with respect to r from 0 to r, one obtains from (4—4)

$$\text{(4—5)} \qquad \sum_{\lambda=1}^{n}\lambda|b_\lambda|^2 r^{2\lambda} < \frac{C(k)}{(1-r_1^2)^{\frac{1}{k}}\cdot(1-r^2)^{\frac{1}{k}}}.$$

On the other hand, we can write by CAUCHY-SCHWARZ inequality

$$\sum_{\lambda=1}^{n}\lambda|b_\lambda|r^{2\lambda-1} \leq \left(\sum_{\lambda=1}^{n}\lambda|b_\lambda|^2 r^{2\lambda}\right)^{\frac{1}{2}}\cdot\left(\sum_{\lambda=1}^{n}\lambda r^{2\lambda-2}\right)^{\frac{1}{2}}$$

from which, considering (4—5) and $\sum_{\lambda=1}^{\infty}\lambda r^{2\lambda-2}=\frac{1}{(r_1^2-r^2)^2}$, we obtain

$$\sum_{\lambda=1}^{n}\lambda|b_\lambda|r^{2\lambda} < \frac{C(k)}{(1-r_1^2)^{\frac{1}{2k}}\cdot(r_1^2-r^2)^{\frac{1}{2k}+1}}.$$

Then, integrating both sides of the foregoing inequality with respect to r from 0 to r, we obtain[1])

[1]) Here, before integration, we replace $r_1^2-r^2$ in the denominator by $r_1(r_1-r)$, and, pass to the following inequality by assuming $r_1 \geq \frac{1}{2}$.

$$\sum_{\lambda=1}^{n} | b_\lambda | r^{2\lambda} < \frac{C(k)}{(1-r_1^2)^{\frac{1}{2k}} \cdot (1-r)^{\frac{1}{2k}}};$$

and conclude by taking $r_1 = \sqrt{r}$

$$\sum_{\lambda=1}^{n} | b_\lambda | < \frac{C(k)}{r^{2n}(1-r)^{\frac{1}{k}}}. \tag{4—6}$$

5. Now, let $g(\zeta) = c_1\zeta + c_2\zeta^2 + \cdots c_n \zeta^n + \cdots$. By comparing the coefficients on both sides, we have from (4—1)

$$n b_n = \sum_{\lambda=0}^{n-1} b_\lambda C_{n-\lambda}. \tag{5—1}$$

Next, let us try to make an estimate for $| c_n |$. Let $\zeta \frac{f'(\zeta)}{f(\zeta)} = 1 + d_1\zeta + \cdots + d_n \zeta^n + \cdots$. By comparing the coefficients on both sides of (4—2), we have

$$c_n = \frac{1}{k} d_n - \frac{1}{\zeta_1^n}.$$

Now, considering the foregoing inequality, we can write from (4—6) and (5—1)

$$| b_n | < \frac{1}{n}\left(\frac{2}{k} + \frac{1}{r^n}\right)\left[\frac{C(k)}{r^{2n}(1-r)^{\frac{1}{k}}} + | b_0 | \right].$$

Further, considering the foregoing inequality, we can write from (3—4) and the Lemma 1—4 [1])

$$\left| | a_{n+1}^{(k)} | - | a_n^{(k)} | \right| < C_1(k)\left(\frac{2}{k} + \frac{1}{r^n}\right) \cdot \frac{1}{n r^n (1-r)^{\frac{1}{k}}} + C_2(k)\, n^{\frac{2}{k}-1}(1-r).$$

Finally, setting $r = 1 - \frac{1}{2n}$ on the right-hand side of the foregoing inequality, one concludes, after some simple reasonings,

$$\left| | a_{n+1}^{(k)} | - | a_n^{(k)} | \right| < C(k)\, n^{\frac{1}{k}-1}$$

as proposed at the beginning of the paper.

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1) It must be noticed in the foregoing inequality that, in view of (3—2),

$$| b_0 | = | \zeta_1 | = r_1 < 1$$

On the other hand, we can write from (3—4)

$$\sum_{\lambda=1}^{n} \left| \, | a_{\lambda+1}^{(k)} | - | a_{\lambda}^{(k)} | \, \right| < \sum_{\lambda=1}^{n} | b_{\lambda} | + (1-r) \sum_{\lambda=1}^{n} | a_{n+1}^{(k)} | \, .$$

Using (4—6) and the Lemma 1—4 on the right-hand side of the foregoing inequality, one obtains

$$\sum_{\lambda=1}^{n} \left| \, | a_{\lambda+1}^{(k)} | - | a_{\lambda}^{(k)} | \, \right| < \frac{C_1(k)}{r^{2n}(1-r)^{\frac{1}{k}}} + C_2(k)\, n^{\frac{2}{k}} \cdot (1-r).$$

Finally, setting $r = 1 - \frac{1}{2n}$ on the right-hand side of the foregoing inequality, one concludes, after some simple reasonings, that

$$\frac{1}{n} \sum_{\lambda=1}^{n} \left| \, | a_{\lambda+1}^{(k)} | - | a_{\lambda}^{(k)} | \, \right| < C(k)\, n^{\frac{1}{k}-1},$$

as proposed at the beginning of the paper.

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ERRATA

* All $\mathfrak{S}^*$'s and $\mathfrak{S}^*_k$'s in 6th and 7th lines from below on page 10 should read respectively $\mathfrak{S}$ and $\mathfrak{S}_k$.
* [4) in 3rd line from above on page 11 should read [4].
* $\operatorname{Re}\left(z^k \frac{f'(z^k)}{f(z^k)}\right) < 0$ on 12th line from below on page 11 should read $\operatorname{Re}\left(z^k \frac{f'(z^k)}{f(z^k)}\right) > 0$.
* The factor $(-\varrho_1^2 k)^{\frac{1}{2k}}$ in the denominator of the right-band side of (2—4) on page 12 should read $(1-\varrho_1^2 k)^{\frac{1}{2k}}$.
* On page 13, (3—2) on 11th line should read (3—3); (3—3) on 12th line from below should read (3—4); (3—4) ou 10th line from below should read (3—5); and (3—3) on 9th line from below should read (3—4).
* The factor $(1-r^2)^{\frac{1}{k}}$ in the denominator of the fraction on the right-hand side of (4—4) on page 14 should read $(r_1^2 - r^2)^{\frac{1}{k}+1}$.
* The factor $(1-r^2)^{\frac{1}{k}}$ in the denominator of the fraction on the right-hand side of (4—5) should read $(r_1^2 - r^2)^{\frac{1}{k}}$.
* The inequality $\sum_{\lambda=1}^{\infty} \lambda\, r^{2\lambda-2} < \frac{1}{(1-r^2)^2}$ in the 4th line from below on page 14 should read $\sum_{\lambda=1}^{\infty} \lambda\, r^{2\lambda-2} < \frac{1}{(r_1^2 - r^2)^2}$.
* The exponent $\frac{1}{2k}$ of the second factor in the denominator of the faction on the right-hand side of the inequality in the 3rd line from blow on page 14 should read $\frac{1}{2k} + 1$.
* The second factor $(1-r)^{\frac{1}{2k}}$ in the denominator of the right-hand side of the inequality on the top of page 15 should read $(r_1 - r)^{\frac{1}{2k}}$.
* $r_1 = \sqrt{r}$ on the second line on the top of page 15 should read $r_1 = \frac{1+r}{2}$.
* The capital C on the right-hand side of (5—1) on page 15 should read small c.
* (3—4) on 7th line from below ou page 15 should read (3—5).

ÖZET

Bu makalesinde yazar

$$f_k(z) = z + a_2^{(k)} z^{k+1} + \cdots + a_n^{(k)} z^{(n-1)k+1} + \cdots$$

şeklindeki yalınkat yıldızıl fonksiyonlar sınıfına münhasır olmak üzere,

$$\left| \, | a_{n+1}^{(k)} | - | a_n^{(k)} | \, \right|$$

büyüklüğünün üst sınırı hakkında evvelce $k=1$ hali için bulunmuş olan bir neticeyi genelleştirmekte, ve keza

$$\frac{1}{n} \sum_{\lambda=1}^{n} \left| \, | a_{\lambda+1}^{(k)} | - | a_{\lambda}^{(k)} | \, \right|$$

büyüklüğünün de bir üst sınırını bulmaktadır.