CERTAIN THEOREMS ON SELF-RECIPROCAL FUNCTIONS

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Let Φ be the LAPLACE transform of f and either of the two functions f or Φ have a given property of self reciprocity in the HANKEL transform: then the object of this paper is to investigate under what conditions the same property will hold for the other function.

1. It is well-known that the LAPLACE transform of $\Phi(p)$ is given by:

(A) $$\Phi(p) = \int_0^\infty e^{-pt} f(t)\, dt.$$

If we multiply by p the right hand side, then symbolically we can write $\Phi(p) \doteq f(t)$ and the symbol $\doteq$ is called «operational».

Let

(i) $$\Phi_1(p) \doteq f_1(t),$$

(ii) $$\Phi_2(p) \doteq f_2(t);$$

then after applying GOLDSTEIN's theorem [2], we get

$$\int_0^\infty f_1(t)\, \Phi_2(t) \frac{dt}{t} = \int_0^\infty \Phi_1(t)\, f_2(t) \frac{dt}{t},$$

where $\Phi_i(p)$ is the image and $f_i(t)$ is the original. Our object is to investigate (i) whether if either of the two functions $f(t)$ and $\Phi(t)$ has a given property of self-reciprocity, does that property hold for the other function and (ii) to ascertain under what conditions this property will hold. Incidentally, we have obtained certain correspondences between the originals and the images,

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when either $f(t)\,t^{\alpha}$ and $f(t)\,t^{\beta}$ are self-reciprocal in the HANKEL transform* [1], α and β having particular values.

2. Theorem 1 : Let

(i) $$f(t) \doteqdot \Phi(\sqrt{p}),$$

(ii) $$K(x) \doteqdot p^{\mu} f(p^2),$$

(iii) $h(x)$ be the HANKEL transform of $t^{2n+\nu-\frac{3}{2}}\,\Phi\left(\frac{t}{2}\right)$, then

(1.1) $$p^{2n+\nu+\mu-\frac{1}{2}}\,h(p) \doteqdot 2^{2n+\nu-1}\,(n)!\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+\frac{\mu}{2}+1}}\,L_n^{\nu}\left(\frac{1}{t}\right) K\left(\frac{x}{\sqrt{t}}\right) dt.$$

Proof : We have

$$f(a^2 t) \doteqdot \left(\frac{\sqrt{p}}{a}\right)$$

and

$$t^{n+\frac{\nu}{2}}\,J_\nu(2\sqrt{t}) \doteqdot \frac{(n)!\,e^{-\frac{1}{p}}}{p^{n+\nu}}\,L_n^{\nu}\left(\frac{1}{p}\right).$$

Applying GOLDSTEIN's theorem, we get

(1.2) $$\int_0^\infty t^{n+\frac{\nu}{2}}\,J_\nu(2\sqrt{t})\,\Phi\left(\frac{\sqrt{t}}{a}\right)\frac{dt}{t} = (n)!\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu}}\,L_n^{\nu}\left(\frac{1}{t}\right) f(a^2 t)\,\frac{dt}{t}.$$

On writing p for a, we have

$$\int_0^\infty t^{n+\frac{\nu}{2}-1}\,J_\nu(2\sqrt{t})\,\Phi\left(\frac{\sqrt{t}}{p}\right) dt = (n)!\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+1}}\,L_n^{\nu}\left(\frac{1}{t}\right) f(p^2 t)\,dt.$$

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* Let

(A) $$g(y) = \int_0^\infty f(x)\sqrt{xy}\,J_\nu(xy)\,dx.$$

In this case $g(y)$ is said to be the HANKEL transform of $f(x)$ of order ν. In particular, if $f(x) \equiv g(x)$, then (A) reduces to

$$f(y) = \int_0^\infty f(x)\sqrt{xy}\,J_\nu(xy)\,dx;$$

$f(x)$ is then said to be self-reciprocal under the HANKEL transform of order ν and is denoted by R_ν.

But $p^{\mu} f(p^2) \doteqdot K(x)$. Therefore

$$(1.3)\quad \int_0^\infty t^{n+\frac{\nu}{2}-1} J_\nu(2\sqrt{t})\, \Phi\left(\frac{\sqrt{t}}{p}\right) dt \doteqdot (n)! \int_0^\infty \frac{e^{-\frac{1}{t}}}{p^{\mu}\, t^{n+\nu+\frac{\mu}{2}+1}} L_n^\nu\left(\frac{1}{p}\right) K\left(\frac{x}{\sqrt{t}}\right) dt$$

or,

$$p^{2n+\nu+\mu-\frac{1}{2}} \int_0^\infty t^{2n+\nu-\frac{3}{2}} \sqrt{pt}\, J_\nu(pt)\, \Phi\left(\frac{t}{2}\right) dt$$

$$\doteqdot 2^{2n+\nu-1} (n)! \int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+\frac{\mu}{2}+1}} L_n^\nu\left(\frac{1}{p}\right) K\left(\frac{x}{\sqrt{t}}\right) dt.$$

But $h(x)$ is the HANKEL transform of $t^{2n+\nu-\frac{3}{2}} \Phi\left(\frac{t}{2}\right)$. Therefore

$$(1.4)\quad p^{2n+\nu+\mu-\frac{1}{2}} h(p) \doteqdot 2^{2n+\nu-1} (n)! \int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+\frac{\mu}{2}+1}} L_n^\nu\left(\frac{1}{p}\right) K\left(\frac{x}{\sqrt{t}}\right) dt.$$

3. Corollary: Let

(*i*) $$f(t) \doteqdot \Phi(\sqrt{p}),$$

(*ii*) $$K(x) \doteqdot p^{\mu} f(p^2),$$

(*iii*) $$t^{2n+\nu-\frac{3}{2}} \Phi\left(\frac{t}{2}\right) \text{ is } R_\nu,$$

then

$$(1.5)\quad p^{4n+2\nu+\mu-2} \Phi\left(\frac{p}{2}\right) \doteqdot 2^{2n+\nu-1} (n)! \int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+\frac{\mu}{2}+1}} L_n^\nu\left(\frac{1}{t}\right) K\left(\frac{x}{\sqrt{t}}\right) dt.$$

4. Example: Let

$$t^{2n+\nu-\frac{3}{2}} \Phi\left(\frac{t}{2}\right) = t^{-\frac{1}{2}},$$

$$\Phi(t) = 2^{1-\nu-2n}\; t^{1-\nu-2n},$$

therefore $$\Phi(\sqrt{p}) = 2^{\frac{1}{2}-\frac{\nu}{2}-n}\; p^{\frac{1}{2}-\frac{\nu}{2}-n}$$

$$\doteqdot \frac{2^{\frac{1}{2}-\frac{\nu}{2}-n}\; t^{n+\frac{\nu}{2}-\frac{1}{2}}}{\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right)} \equiv f(t)$$

$$\text{so} \quad p^{\mu} f(p^2) = \frac{2^{1-\nu-2n} \quad p^{2n+\nu+\mu-1}}{\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right)}$$

$$\doteqdot \frac{2^{1-\nu-2n} \quad x^{1-\nu-\mu-2n}}{\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right) \quad \Gamma(2-\mu-\nu-2n)} \equiv K(x).$$

But the Hankel transform of $t^{-1/2}$ is $p^{-1/2}$. Hence from (1.4) we have

$$p^{2n+\nu+\mu-1} \doteqdot \frac{(n)! \quad x^{1-\nu-\mu-2n}}{\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right) \quad \Gamma(2-\mu-\nu-2n)} \int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{\frac{\nu+3}{2}}} \; L_n^{\nu}\left(\frac{1}{t}\right) dt$$

or,

$$\text{(1.6)} \qquad \int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{\nu+\frac{3}{2}}} \; L_n^{\nu}\left(\frac{1}{t}\right) dt = \frac{\Gamma\left(n+\frac{\nu}{2}+\frac{1}{2}\right)}{(n)!}.$$

5. Example for the corollary :

$$x^{\nu-2m-\frac{1}{2}} \; e^{\frac{x^2}{4}} \; W_{k,m}\left(\frac{x^2}{2}\right) \textit{ is } R_{\nu}, \qquad \text{where } k = 3m-\nu-\tfrac{1}{2};$$

$$\text{then} \quad \Phi(\sqrt{p}) = 2^{1-2n-2m} \; p^{\frac{1}{2}-n-m} \; e^{p} \; W_{3m-\nu-\frac{1}{2},\, m}(2p)$$

Suppose $n=0$, then

$$\Phi(\sqrt{p}) = 2^{1-2m} \; p^{\frac{1}{2}-m} \; e^{p} \; W_{3m-\nu-\frac{1}{2},\, m}(2p).$$

$$\doteqdot \frac{t^{\nu-2m}(2+t)^{4m-\nu-1}}{2^{3m-\frac{3}{2}} \; \Gamma(\nu-2m+1)} \equiv f(t)$$

$$p^{\mu} f(p^2) = \frac{p^{\mu+2\nu-4m}\,(2+p^2)^{4m-\nu-1}}{2^{3m-\frac{3}{2}} \; \Gamma(\nu-2m+1)}$$

$$\doteqdot \frac{x^{2-\mu-4m}}{2^{3m-\frac{3}{2}} \; \Gamma(\nu-2m+1) \; \Gamma(3-\mu-4m)} \, {}_1F_2\left(\nu-4m+1;\; \frac{3-\mu}{2}-2m,\; 2-\frac{\mu}{2}-2m;\; \frac{-x^2}{2}\right) \equiv K(x),$$

(1.7)

$$p^{2\nu+2n+\mu-2m-1} \; e^{\frac{p^2}{4}} \; W_{3m-\nu-\frac{1}{2},\, m}\left(\frac{p^2}{2}\right) \doteqdot \frac{2^{2n+\nu-3m+\frac{1}{2}}\,(n)!}{\Gamma(\nu-2m+1) \; \Gamma(3-\mu-4m)}$$

$$\times\ x^{2-\mu-4m}\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu-m+2}}\, L_n^{\nu}\left(\frac{1}{t}\right)\, {}_1F_2\left(\nu-4m+1\,;\ \frac{3-\mu}{2}-2m,\ 2-\frac{\mu}{2}-2m\,;\ -\frac{x^2}{2}\right)dt$$

where n is a positive integer ($n\neq 0$). In particular, if $\mu=2$, $m=-1/4$, then

(1.8)

$$p^{4n+2\nu+\frac{3}{2}}\, e^{\frac{p^2}{4}}\, W_{-\frac{1}{4},\frac{1}{4}}(2p) \doteqdot \frac{2^{2n+\nu+\frac{3}{4}}}{\sqrt{\pi}}\,(n\,!)\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{n+\nu+2}}\, L_n^{\nu}\left(\frac{1}{t}\right)\sin\left(x\sqrt{\frac{2}{t}}\right)dt$$

where n is a positive integer ($n\geq 0$).

6. **Note :** If we take $n=0$ in Theorem 1, we will get the result in place of (1.1) as

$$p^{\mu+\nu-\frac{1}{2}}\, h(p) \doteqdot 2^{\nu-1}\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{\nu+\frac{\mu}{2}+1}}\, K\left(\frac{x}{\sqrt{t}}\right)dt, \tag{1.9}$$

and the corollary will turn out to be as follows :

Let

(*i*) $$f(t) \doteqdot \Phi(\sqrt{p}),$$

(*ii*) $$K(x) \doteqdot p^{\mu}\,\Phi(p^2),$$

(*iii*) $$t^{\nu-\frac{3}{2}}\,\Phi\left(\frac{t}{2}\right)\ \text{be}\ R_{\nu},$$

then

$$p^{2\nu+\mu-2}\,\Phi\left(\frac{p}{2}\right) \doteqdot 2^{\nu-1}\int_0^\infty \frac{e^{-\frac{1}{t}}}{t^{\nu+\frac{\mu}{2}+1}}\, K\left(\frac{x}{\sqrt{t}}\right)dt.$$

Again let

$$t^{\nu+\frac{\mu}{2}-3}\, K(\sqrt{t}) \doteqdot \psi(p)$$

then

$$p^{2\nu+\mu-2}\,\Phi\left(\frac{p}{2}\right) \doteqdot \frac{2^{\nu-1}}{x^{2\nu+\mu-2}}\,\psi\left(\frac{1}{x^2}\right). \tag{1.10}$$

7. Theorem 2 : Let

(i) $$t^m f(t) \doteqdot \Phi(p),$$

(ii) $$h(x) \doteqdot p^\mu \Phi\left(\frac{1}{p}\right),$$

(iii) $K(x)$ be the HANKEL transform of $t^{-m-\frac{3}{2}} f\left(\frac{2}{t}\right)$,

then

(2.1) $$p^{\mu-\frac{1}{2}} K(p) \doteqdot 2^{-m} \int_0^\infty t^{\mu-1} J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) h(xt)\, dt.$$

Proof: We have

$$a^m t^m f(at) \doteqdot \Phi\left(\frac{p}{a}\right)$$

and

$$J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) \doteqdot J_\nu\left(\frac{2}{p}\right).$$

Applying GOLDSTEIN's theorem, we get

(2.2) $$a^m \int_0^\infty t^m J_\nu\left(\frac{2}{t}\right) f(at) \frac{dt}{t} = \int_0^\infty J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) \Phi\left(\frac{t}{a}\right) \frac{dt}{t}.$$

Replacing a by p, we have

$$p^m \int_0^\infty t^{m-1} J_\nu\left(\frac{2}{t}\right) f(pt)\, dt = \int_0^\infty \frac{1}{t} J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) \Phi\left(\frac{t}{p}\right) dt.$$

But $h(x) \doteqdot p^\mu \Phi\left(\frac{1}{p}\right)$. Therefore

$$p^m \int_0^\infty t^{m-1} J_\nu\left(\frac{2}{t}\right) f(pt)\, dt \doteqdot \int_0^\infty t^{\mu-1} J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) h(xt)\, dt$$

or,

(2.3) $$p^{\mu-\frac{1}{2}} \int_0^\infty t^{-m-\frac{3}{2}} \sqrt{pt}\, J_\nu(pt) f\left(\frac{2}{t}\right) dt \doteqdot 2^{-m} \int_0^\infty J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t})\, t^{\mu-1} h(xt)\, dt.$$

But $h(x)$ is the HANKEL transform of $t^{-m-3/2} f(2/t)$. Therefore

(2.4) $$p^{\mu-\frac{1}{2}} K(p) \doteqdot 2^{-m} \int_0^\infty t^{\mu-1} J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) h(xt)\, dt$$

8. Corollary: Let

(i) $$t^m f(t) \doteqdot \Phi(p),$$

(ii) $$h(x) \doteqdot p^\mu \Phi\left(\frac{1}{p}\right),$$

(iii) $$t^{-m-\frac{3}{2}} f\left(\frac{2}{t}\right) \text{ be } R_\nu,$$

then

(2.5) $$p^{\mu-m-2} \Phi\left(\frac{2}{p}\right) \doteqdot 2^{-m} \int_0^\infty t^{\mu-1} J_\nu(2\sqrt{t}) I_\nu(2\sqrt{t}) h(xt)\, dt.$$

9. Example: Suppose

$$t^{-m-\frac{3}{2}} f\left(\frac{2}{t}\right) = t^{\nu+\frac{1}{2}} e^{-t},$$

therefore

$$t^m f(t) = \frac{2^{\nu+m+2}}{t^{\nu+2}} e^{-\frac{2}{t}}.$$

Let $\nu = -\frac{1}{2}$, therefore

$$t^m f(t) = 2^{m+\frac{3}{2}} t^{-\frac{3}{2}} e^{-\frac{2}{t}} \doteqdot 2^{m+\frac{3}{2}} \sqrt{\frac{\pi}{2}}\, p\, e^{-2\sqrt{2p}} \equiv \Phi(p);$$

$$p^\mu \Phi\left(\frac{1}{p}\right) = 2^{m+1} \sqrt{\pi}\, p^{\mu-1} e^{-\frac{2\sqrt{2}}{p}}$$

$$\doteqdot \frac{2^{m+1}}{2^{2\mu+\frac{3}{2}} x^{\mu+1}} \int_0^\infty e^{-\frac{t^2}{4x}} He_{2\mu}\left(\frac{t}{2\sqrt{x}}\right) J_2\left(2\sqrt{\sqrt{2}\, t}\right) t^\mu\, dt \equiv h(x).$$

The HANKEL transform of $t^{\nu+\frac{1}{2}} e^{-t}$ is $\sqrt{\frac{2}{\pi}} \frac{1}{(p^2+1)^2}$. So

$$(2.6)\qquad \sqrt{\frac{2}{\pi}}\,\frac{p^{\mu-\frac{1}{2}}}{(p^2+1)^2} \doteq \frac{1}{2^{2\mu+\frac{1}{2}}\,x^{\mu+1}} \int_0^\infty \frac{1}{t^2}\, J_{-\frac{1}{2}}(2\sqrt{t})\, I_{-\frac{1}{2}}(2\sqrt{t}) \left[\int_0^\infty e^{-\frac{t}{4x}} \times He_{2\mu}\left(2\sqrt{\frac{t}{x}}\right) J_2\left(2\sqrt{2\sqrt{2}}\,t\right) t^{\mu}\, dt \right] dt$$

or,

$$\frac{2^{2\mu+1}\, x^{\frac{11}{2}-2\mu}}{\sqrt{x}\,\Gamma\left(\frac{11}{2}-\mu\right)}\, {}_1F_2\left(2\,;\,\frac{11}{4}-\frac{\mu}{2},\ \frac{13}{4}-\frac{\mu}{2}\,;\,-\frac{x^2}{4}\right)$$

$$= \int_0^\infty \frac{1}{t^2}\, J_{-\frac{1}{2}}(2\sqrt{t})\, I_{-\frac{1}{2}}(2\sqrt{t}) \left[\int_0^\infty e^{-\frac{t}{4x}}\, He_{2\mu}\left(2\sqrt{\frac{t}{x}}\right) \times J_2\left(2\sqrt{2\sqrt{2}}\,t\right) t^{\mu}\, dt \right] dt.$$

10. Theorem 3: Let

(*i*) $$f(t) \doteq p^{\mu}\,\Phi(p),$$

(*ii*) $$K'(x) \doteq p^m f\left(\frac{1}{p}\right),$$

(*iii*) $h(x)$ be the HANKEL transform of $t^{-\mu-\frac{1}{2}}\,\Phi\left(\frac{2}{t}\right)$,

then

$$p^{m+\frac{1}{2}}\, h(p) \doteq 2 \int_0^\infty t^m J_\nu(\sqrt{t})\, K_\nu(\sqrt{t})\, K'(xt)\, dt.$$

11. Corollary: Let

(*i*) $$f(t) \doteq p^{\mu}\,\Phi(p),$$

(*ii*) $$K'(x) \doteq p^m f\left(\frac{1}{p}\right),$$

(*iii*) $t^{-\mu-\frac{1}{2}}\,\Phi\left(\frac{2}{t}\right)$ be R_ν,

then

$$p^{m-\mu}\,\Phi\left(\frac{2}{p}\right) \doteq 2^{1-\mu} \int_0^\infty t^m J_\nu(\sqrt{t})\, K_\nu(\sqrt{t})\, K'(xt)\, dt.$$

12. Theorem 4 : Let

(*i*) $$f(t) \doteqdot \Phi(p),$$

(*ii*) $$K(x) \doteqdot p^{\mu} f(p^2),$$

(*iii*) $$t^{2\mu-\frac{5}{2}} \Phi\left(\frac{t^2}{4}\right) \text{ be } R_{2\nu},$$

then

$$p^{5\mu-4} \Phi\left(\frac{p^2}{4}\right) \doteqdot \frac{2^{2\nu-2} \Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)} \int_0^\infty \frac{e^{-\frac{1}{2t}}}{t^{3\mu/2}} M_{\mu,\nu}\left(\frac{1}{t}\right) K\left(\frac{x}{\sqrt{t}}\right) dt.$$

13. Theorem 5 : Let

(*i*) $$f(t) \doteqdot \Phi(p^2),$$

(*ii*) $$h(x) \doteqdot p^{\mu} f(\sqrt{p}),$$

(*iii*) $$t^{\frac{\mu}{2}-1} \Phi(2t) \text{ be } R_{\nu+\frac{1}{2}},$$

then

$$p^{\mu-\frac{3}{2}} \Phi(2p) \doteqdot \frac{\Gamma(\nu+1)}{2^{\frac{\mu}{2}-2}\sqrt{\pi}} \int_0^\infty D_{-\nu-1}(t\, e^{i\pi/4}) D_{-\nu-1}(t\, e^{-i\pi/4})\, h\left(\frac{x}{t}\right) \frac{dt}{t^{2m}}.$$

Proofs of theorems 3, 4 and 5 : By taking

$$\frac{1}{t} J_\nu\left(\frac{2}{t}\right) \doteqdot p\, J_\nu(\sqrt{p})\, K_\nu(\sqrt{p}),$$

$$t^{\mu-\frac{1}{2}} J_\nu(2\sqrt{t}) \doteqdot \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)} e^{-\frac{1}{2p}} p^{1-\mu} M_{\mu,\nu}\left(\frac{1}{p}\right),$$

and

$$J_{\nu+\frac{1}{2}}\left(\frac{t^2}{2}\right) \doteqdot \frac{\Gamma(\nu+1)}{\sqrt{\pi}} p\, D_{-\nu-1}(p\, e^{i\pi/4}) D_{-\nu-1}(p\, e^{-i\pi/4})$$

in theorems 3, 4 and 5 we can prove these theorems on following the procedure as adopted in Theorem 1.

REFERENCES

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ÖZET

f fonksiyonunun LAPLACE dönüşüğü φ olsun ve *f* ile φ fonksiyonlarından herhangi birisi, HANKEL dönüştürümüne göre kendi kendisinin karşıtı olma özeliğini haiz bulunsun. Bu takdirde hangi şartlar altında aynı özeliğin diğer fonksiyon için de vârid olabileceği, bu araştırmanın konusunu teşkil etmektedir.