ON APPROXIMATE SEMI-CONTINUOUS FUNCTIONS AND ALLIED SETS

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In this paper approximately semi-continuous functions have been defined and the concepts of metrically closed and open sets have been introduced. Some properties of approximately semi-continuous functions have been established and the behaviour of approximately semi-continuous functions in regard to metrically closed sets and metrically open sets have been studied.

1. Introduction : Let $f(x)$ be a bounded real function defined in the interval $[a, b]$. Let $\xi \in [a, b]$ and $\varepsilon > 0$ be arbitrary. We say that $f(x)$ is *approximately upper semi-continuous* or in short *a. u. s. c.* at $x = \xi$ if there exists a measurable set $S \subset [a, b]$ with ξ as a point of density [1] such that

$$f(x) < f(\xi) + \varepsilon, \quad \text{if} \quad x \in S.$$

Similarly, $f(x)$ is said to be *approximately lower semi-continuous* or in short *a. l. s. c.* at $x = \xi$ if there exists a measurable set $S \subset [a, b]$ with ξ as a point of density such that

$$f(x) > f(\xi) - \varepsilon, \quad \text{if} \quad x \in S.$$

If $f(x)$ is *a. u. s. c.* (or *a. l. s. c.*) at each ξ of $[a, b]$, then it is said to be *a. u. s. c.* (or *a. l. s. c.*) in $[a, b]$.

It is obvious that if $f(x)$ is approximately continuous {[2], p. 132} at $x = \xi$, then it is both *a. l. s. c.* and *a. u. s. c.* at $x = \xi$ and also conversely. But there exist functions which are neither upper semi-continuous nor approximately continuous at a point ξ where it may be *a. u. s. c.*

The following is an illustration :

$$\begin{aligned} f(x) &= 0 \quad \text{if} \quad x \text{ is irrational} \\ &= \tfrac{1}{2} \quad \text{if} \quad x = \tfrac{1}{2} \\ &= 1 \quad \text{for all other rationals.} \end{aligned}$$

Here $\xi = \frac{1}{2}$.

The purpose of the present paper is to prove some properties of approximately semi-continuous functions. To do this, we are led to introduce the concept of metrically closed sets and as a consequence we have investigated to what extent approximately continuous functions can be characterised with the aid of the sets complementary to metrically closed sets.

Throughout the paper, we shall always consider those sets which are subsets of the segment $[a, b]$ and those real functions which are single-valued, bounded and defined in $[a, b]$.

2. Theorem 1. *Suppose $f(x)$ and $g(x)$ are a. l. s. c. at $x_0 \in [a, b]$. Then $f(x) + g(x)$ is also a. l. s. c. at x_0.*

Proof : Let A be any number with $f(x_0) + g(x_0) > A$. Then there exist B and C such that $f(x_0) > B$, $g(x_0) > C$, $B + C > A$. Since $f(x)$ and $g(x)$ are *a. l. s. c.* at x_0, by definition there exist measurable sets S_1 and S_2 with x_0 as point of density and such that $f(x) > B$ for $x \in S_1$ and $g(x) > C$ for $x \in S_2$. Let $S = S_1 \cap S_2$. Then S is measurable and x_0 is a point of density of S and $f(x) + g(x) > B + C > A$ for $x \in S$. Since A is any number with $A < f(x_0) + g(x_0)$ the theorem follows.

Theorem 2. *Let an increasing sequence of functions*

$$f_1(x) \leq f_2(x) \leq \cdots\cdots \leq f_n(x) \leq \cdots\cdot$$

be given ; suppose that all the functions $f_n(x)$ are a. l. s. c. at $x_0 \in [a, b]$. Then $f(x) = \underset{n\to\infty}{Lt} f_n(x)$ is a. l. s. c. at x_0.

Proof : Let A be a real number such that $f(x_0) > A$. Then for all sufficiently large values of n, $f_n(x_0) > A$. We choose one of the values of these n's, say $n = m$ and keep it fixed. Since $f_m(x)$ is *a. l. s. c.* at $x = x_0$ it follows that there exists a measurable set $S \subset [a, b]$ with x_0 as point of density and such that $f_m(x) > A$ if $x \in S$. Since for every x, $f(x) \geq f_m(x)$ we have $f(x) > A$ if $x \in S$. As A is any number with $A < f(x_0)$, the theorem follows.

Combining theorems 1 and 2 we get the following :

Theorem 3. *Suppose that all the terms of the series $\Sigma u_n(x)$ ore nonnegative and the series converges in $[a, b]$. If each $u_n(x)$ is a. l. s. c. at x_0, then the sum of the series is also a. l. s. c. at x_0.*

Definition : A point α is said to be a metric limit point of a set S, if in every neighbourhood of α, there is a point of density of S. A set S is said to be *metrically closed* if it contains all its metric limit points.

Theorem 4. *If A is any set, then the set of metric limit points of A is metrically closed.*

Proof: Let A_1 denote the set of metric limit points of A and let ξ be a metric limit point of A_1. We are to show that $\xi \in A_1$. Let I_ξ be a neighbourhood of ξ. Since ξ is a metric limit point of A_1, there exists a point of density of A_1 and consequently a point of density A in I_ξ. Since I_ξ is any neighbourhood of ξ, ξ is a metric limit point of A. Hence $\xi \in A_1$ and the theorem is proved.

Theorem 5. *If for any real number α the set*

$$E\{x \in [a, b]; \quad f(x) \geq \alpha\}$$

is metrically closed, then $f(x)$ is a. u. s. c. in $[a, b]$.

Proof: Let $\varepsilon > 0$ be arbitrary and ξ be taken at random on the segment $[a, b]$.

Let

$$S_1 = E\{x \in [a, b]; \quad f(x) \geq f(\xi) + \varepsilon\}$$

and

$$S_2 = E\{x \in [a, b]; \quad f(x) < f(\xi) + \varepsilon\}.$$

So, $S_1 \cup S_2 = [a, b]$ and $S_1 \cap S_2 = \Phi$. Since S_1 is, by hypothesis, metrically closed and $\xi \in S_2$, there exists a neighbourhood of ξ almost all points of which belong to S_2. Hence there exists a measurable set S with ξ as a point of density and such that

$$f(x) < f(\xi) + \varepsilon \quad \text{if} \quad x \in S.$$

Hence $f(x)$ is *a. u. s. c.* at $x = \xi$. Since ξ is any point in $[a, b]$ the theorem follows.

Note: It is obvious that theorems 1, 2 and 5 have analagous counterparts if one replaces *a. l. s. c.* by *a. u. s. c.* and viceversa.

3. If S is any metrically closed set in $[a, b]$ and $\xi \in S$. then it follows that there exists a neighbourhood of ξ almost all points of which belong to the complement of S. This consideration suggests the following definition.

Definition: A point ξ of a set S is said to be a ***metric interior point*** of S, if there exists a neighbourhood of ξ almost all points of which are points of S. A set S is said to be ***metrically open*** if every point of S is a metric interior point of S.

The following results follow immediately:

(1) If A is a metrically closed set, then its complement is metrically open.

(2) If A is a metrically open set, then its complement is metrically closed.

(3) If A and B are metrically open sets, then $A \cup B$ and $A \cap B$ are metrically open sets.

(4) Union of an arbitrary family of metrically open sets is a metrically open set.

4. It may be of some interest to ask what relations these metrically open sets bear with the approximately continuous functions. In the following we have attempted to establish some of these connections. To do this, we introduce another type of set defined below.

Definition : A set S is said to be an A-type set if every point of S is a point of density of S.

Theorem 6. *Let $f(x)$ be a bounded function defined in $[a, b]$. If for every metrically open set O, $f^{-1}(O)$ is metricolly open, then $f(x)$ is approximately continuous in $[a, b]$.*

Conversely, if $f(x)$ is approximately continuous and f^{-1} fulfils the condition (N) {[2], p. 224} in $[a, b]$ then for any metrically open set O, $f^{-1}(O)$ is an A-type set.

Proof : Let $\xi \in [a, b]$ and $\varepsilon > 0$ be arbitrary. Let O be a metrically open set with span less than ε and containing $f(\xi)$. Then $\xi \in \xi^{-1}(O)$. Since, by hypothesis, $f^{-1}(O)$ is metrically open and $\xi \in f^{-1}(O)$ there exists a measurable set S with ξ as a point of density and $| f(x) - f(\xi) | < \varepsilon$ if $x \in S$. So, $f(x)$ is approximately continuous at $x = \xi$. Since ξ is any point in $[a, b]$, the first part of the theorem follows.

Conversely, let $f(x)$ be approximately continuous in $[a, b]$. Let O be a metrically open set. If $f^{-1}(O)$ is void, the theorem is proved. So, let $\xi \in [a, b]$ and $f(\xi) \in O$. Since $f(x)$ is approximately continuous at $x = \xi$ and f^{-1} fulfils the condition (N) in $[a, b]$, there exists a set A_ξ with ξ as a point of density and $f(A_\xi) \subset O$. Let the point ξ be adjoined to the set A_ξ and the resulting set denoted by $\bar{A}_\xi$.

Then
$$\xi \in \bar{A}_\xi \subset f^{-1}(O).$$

So,
$$f^{-1}(O) = \bigcup_{\xi \in f^{-1}(O)} \bar{A}_\xi.$$

Hence every point of $f^{-1}(O)$ is a point of density of $f^{-1}(O)$, i.e. $f^{-1}(O)$ is an A—type set.

This completes the proof.

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ÖZET

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