ON THE DERIVATIVES OF AN INTEGRAL FUNCTION REPRESENTED BY DIRICHLET SERIES

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The object of this paper is to examine the properties of the maximum term of the derivatives of an integral function represented by a DIRICHLET series in the whole plane. Some results concerning the maximum modulus and an expression for the derivative of the maximum term have also been obtained.

1. Consider the DIRICHLET series

(1,1) $$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n},$$

where

$$\lambda_{n+1} > \lambda_n, \ \lambda_1 > 0, \ \lim_{n\to\infty} \lambda_n = \infty, \ s = \sigma + it$$

and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0.$$

Let σ_c and σ_a be the abscissae of convergence and absolute convergence of $f(s)$ respectively. Let $f(s)$ represent an entire function with $\sigma_c = \sigma_a = \infty$.

Let

$$M(\sigma, f) = \underset{-\infty<t<\infty}{\text{l. u. b.}} | f(\sigma + it) | , \ \mu(\sigma, f) = \max_{n\geq 1} | a_n | e^{\sigma\lambda_n} .$$

If $\nu(\sigma, f)$ denotes the values of n, for which $\mu(\sigma, f) = | a_n | e^{\sigma\lambda_n}$, we call it the rank of the maximum term $\mu(\sigma, f)$. If there are more than one such values of n, we consider as rank the greatest of them. The type and lower type of (1.1) are defined as the superior and inferior limits respectively of

$e^{-\varrho\sigma} \log M(\sigma, f)$ as $\sigma \to \infty$, where ϱ $(0 < \varrho < \infty)$ is the linear order of $f(s)$.

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Y. C. YU [5] has proved that if the linear order of $f(s)$ is finite and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0,$$

then for large σ

$$\log M(\sigma, f) \sim \log \mu(\sigma, f). \tag{1.2}$$

Here we have obtained a few properties of the maximum term of the derivatives of an integral function represented by DIRICHLET series in the whole plane and certain relations with the maximum modulus. We have also determined $\mu^{(1)}(\sigma, f^{(m)})$ in terms of $\mu(\sigma, f^{(m)})$ and $\lambda_\nu(\sigma, f)$ where $\mu^{(1)}(\sigma, f^{(m)})$ is the derivative of $\mu(\sigma, f^{(m)})$, the maximum term of $f^{(m)}(s)$, the m^{th} derivative of $f(s)$. *Throughout this paper we assume that*

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0$$

and

$$\int_{\sigma_0}^{\sigma} t \cdot d\lambda_\nu(t, f^{(m)}) = o(e^{\varrho\sigma}),$$

for $m = 0, 1, 2, \ldots$

2. Theorem 1. *If*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

is an integral function of linear order $\varrho\,(0 < \varrho < \infty)$, *type* T *and lower type* t $(0 \leq t \leq T \leq \infty)$, *then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} = \begin{matrix}T\\ t\end{matrix}, \tag{2.1}$$

where $\sigma \to \infty$ *through values outside a set of measure zero.*

The proof depends on the following lemmas:

Lemma 1. *Let*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

be an integral function of linear order $\varrho\,(0 < \varrho < \infty)$, *type* T *and lower type* t, *then*

(2.2) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\sigma\, e^{-\varrho\sigma} \lambda_{\nu(\sigma, f)}\right) = \frac{T}{t},$$

where $\sigma \to \infty$ through values outside a set of measure zero.

Proof. Y. C. Yu [5] has shown that, if

$$\lim_{n\to\infty} \sup \frac{\log \lambda_n}{\lambda_n} = D < \infty$$

(2.3) $$\log \mu(\sigma, f) = \log \mu(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t, f)}\, dt.$$

Integrating $\lambda_{\nu(t, f)}$ by parts, we get, from (2.3)

$$\log \mu(\sigma, f) = \log \mu(\sigma_0, f) + \sigma \lambda_{\nu(\sigma, f)} - \sigma_0 \lambda_{\nu(\sigma_0, f)} - \int_{\sigma_0}^{\sigma} t \,.\, d\lambda_{\nu(t, f)}.$$

Hence

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\sigma\, e^{-\varrho\sigma} \lambda_{\nu(\sigma, f)}\right) = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(e^{-\varrho\sigma} \log \mu(\sigma, f)\right)$$

$$= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(e^{-\varrho\sigma} \log M(\sigma, f)\right) = \frac{T}{t},$$

provided $\sigma \to \infty$ through values outside a set of measure zero,

$$\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0 \qquad \text{and} \qquad \int_{\sigma_0}^{\sigma} t \,.\, d\lambda_{\nu(t, f)} = 0\,(e^{\varrho\sigma}).$$

Lemma 2. *The type and lower type of the derivative of an integral function are the same as that of the function.*

Proof: In a previous paper [1] I have proved

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma^{-1} \log\left(\frac{M(\sigma, f^{(m)})}{M(\sigma, f^{(m-1)})}\right)\right\} = \frac{\varrho}{\lambda},$$

where λ is the linear lower order of $f(s)$.

Thus, for $\sigma > \sigma_0$, we have

$$\sigma\, e^{-\varrho\sigma} (\lambda - \varepsilon) < e^{-\varrho\sigma} \log M(\sigma, f^{(m)}) - e^{-\varrho\sigma} \log M(\sigma, f^{(m-1)}) < \sigma\, e^{-\varrho\sigma} (\varrho + \varepsilon).$$

or,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{e^{-\varrho\sigma} \log M(\sigma, f^{(m)})\right\} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{e^{-\varrho\sigma} \log M(\sigma, f^{(m-1)})\right\},$$

which proved the lemma since the linear order of $f^{(m)}(s)$ is the same as $f^{(m-1)}(s)$ and $0 < \varrho < \infty$.

Proof of Theorem 1. R. S. L. SRIVASTAVA [8] has obtained the following result.

$$\lambda_{\nu}(\sigma, f) \leq \frac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)} \leq \lambda_{\nu}(\sigma, f^{(1)}) \leq \cdots\cdots\cdot \tag{2.4}$$

From the above result it can easily be shown that

$$\lambda_{\nu}(\sigma, f) \leq \left\{\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right\}^{1/m} \leq \lambda_{\nu}(\sigma, f^{(m)}).$$

Therefore,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\sigma e^{-\varrho\sigma} \lambda_{\nu}(\sigma, f)\right) \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} \leq$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma e^{-\varrho\sigma} \lambda_{\nu}(\sigma, f^{(m)})\right\}.$$

Hence, on using Lemma 1 and Lemma 2, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} = \frac{T}{t},$$

where $\sigma \to \infty$ through values outside a set of measure zero.

Corollary 1.

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f^{(m-1)})}\right)\right\} = \frac{T}{t},$$

where $\sigma \to \infty$ through values outside a set of measure zero.

Corollary 2. For almost all values of $\sigma > \sigma_0$.

$$\sigma^{-m} e^{m\varrho\sigma} (t-\varepsilon)^m \mu(\sigma, f) < \mu(\sigma, f^{(m)}) < \sigma^{-m} e^{m\varrho\sigma} (T+\varepsilon)^m \mu(\sigma, f).$$

Corollary 3. If $t > 0$, then

$$\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)} \to \infty, \quad \text{as} \quad \sigma \to \infty.$$

Corollary 4. If $t > 0$, then the sequence

$$\mu(\sigma, f),\ \mu(\sigma, f^{(1)}),\ \mu(\sigma, f^{(2)}), \ldots\ldots$$

forms an increasing sequence for $\sigma > \sigma_0$.

Corollary 5. If $0 < t \leqq T < \infty$, then

$$\lim_{\sigma\to\infty} \left\{\sigma^{-1} \log\left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} = \varrho,$$

where $\sigma \to \infty$ through values outside a set of measure zero.

This follows from Corollary 2, on taking the logarithm on hoth sides, dividing hy σ and then proceeding to limit.

Application. The integral function

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n},$$

of linear order $\varrho\ (0 < \varrho < \infty)$ is of perfectly linear regular growth $T > 0$, if and only if,

$$\left\{\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right\}^{1/m} \sim \sigma^{-1} e^{\varrho\sigma} T,$$

for large σ, where $\sigma \to \infty$ through values outside a set of measure zero.

Let $f(s)$ be of linear order ϱ and perfectly linear regular growth T, then we have from (2.1)

$$\lim_{\sigma\to\infty} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} = T$$

or,

$$\left\{\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right\}^{1/m} \sim \sigma^{-1} e^{\varrho\sigma} T. \tag{2.5}$$

Again, if (2.5) holds, we have

$$\lim_{\sigma\to\infty} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right)^{1/m}\right\} = T$$

and from (2.1), $f(s)$ is of perfectly linear regular growth T. Now,

(*i*) if the type of the function is one, then

$$\left\{\frac{\mu(\sigma, f^{(m)})}{\mu(\mu, f)}\right\}^{1/m} \sim o\,(\sigma^{-1} e^{\varrho\sigma}),$$

for large σ;

(*ii*) if the type of the function is zero, then

$$\left\{\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)}\right\}^{1/m} = o(\sigma^{-1} e^{\varrho\sigma}) \tag{2.6}$$

and if the function is of non-zero type, then o is replaced by O in (2.6).

3. Determination of $\mu^{(1)}(\sigma, f^{(m)})$.

We have, from Theorem 1, for $\sigma > \sigma_0$

$$(t-\varepsilon)^m e^{m\varrho\sigma} \mu(\sigma, f) < \sigma^m \mu(\sigma, f^{(m)}) < (T+\varepsilon)^m e^{m\varrho\sigma} \mu(\sigma, f),$$

where

$$0 < t \leq T < \infty.$$

Therefore,

$$\sigma^m \mu(\sigma, f^{(m)}) = e^{m\varrho\sigma} \mu(\sigma, f)\left\{(t-\varepsilon)^m + \vartheta\,(T+\varepsilon)^m - (t-\varepsilon)^m)\right\},$$

where

$$0 < \vartheta < 1.$$

Hence, on using the result (2.3), we get

$$m \log\sigma + \log\mu(\sigma_0, f^{(m)}) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t, f^{(m)})}\, dt = m\varrho\sigma + \log\mu\,\sigma_0, f) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t, f)}\, dt$$

$$+ \log\left\{(t-\varepsilon)^m + \vartheta\left((T+\varepsilon)^m - (t-\varepsilon)^m\right)\right\},$$

or

$$\int_{\sigma_0}^{\sigma} \left\{\lambda_{\nu(t, f)^{(m)}} - \lambda_{\nu(t, f)}\right\} dt = m\varrho\sigma - m\log\sigma + C, \tag{3.1}$$

where

$$C = \log\left\{(t-\varepsilon)^m + \vartheta\left((T+\varepsilon)^m - (t-\varepsilon)^m\right)\right\} + \log\mu(\sigma_0, f) - \log\mu(\sigma_0, f^{(m)}),$$

$$0 < \vartheta(\sigma) < 1.$$

Differentiating (3.1), we get

$$\lambda_{\nu(\sigma, f)^{(m)}} - \lambda_{\nu(\sigma, f)} = m\varrho - \frac{m}{\varrho} + \varphi(\sigma),$$

almost everywhere, where, $\varphi(\sigma)$ is bounded and tends to zero as $\sigma \to \infty$. From (2.3), after replacing $f(s)$ by $f^{(m)}(s)$ we, therefore get

$$\mu^{(1)}(\sigma, f^{(m)}) = \mu(\sigma, f^{(m)})\left\{\lambda_{\nu(\sigma, f)} + m\varrho - \frac{m}{\sigma} + \varphi(\sigma)\right\},$$

almost everywhere.

Corollary 1*.

(3.2) $$\lim_{\sigma\to\infty}\left\{\lambda_{\nu}(\sigma, f^{(m)}) - \lambda_{\nu}(\sigma, f)\right\} = m\varrho,$$

where $\sigma \to \infty$ through values outside a set of measure zero.

Corollary 2.

(3.3) $$\lim_{\sigma\to\infty}\left\{\log\lambda_{\nu}(\sigma, f^{(m)}) - \log\lambda_{\nu}(\sigma, f)\right\} = 0,$$

where $\sigma \to \infty$ through values outside a set of measure zero.

Corollary 3.** For almost all values of $\sigma > \sigma_0 \geq 0$.

$$\mu^{(1)}(\sigma, f) < \mu^{(1)}(\sigma, f^{(1)}) < \mu^{(1)}(\sigma, f^{(2)}) < \ldots\ldots\ldots\ldots,$$

provided the lower order λ of $f(s)$ is greater than zero.

Using (2.3) for the functions $f^{(m)}(s)$ and $f^{(m-1)}(s)$, we have

(3.4) $$\log\mu(\sigma, f^{(m)}) = \log\mu(\sigma_0, f^{(m)}) + \int_{\sigma_0}^{\sigma}\lambda_{\nu}(t, f^{(m)})\,dt.$$

(3.5) $$\log\mu(\sigma, f^{(m-1)}) = \log\mu(\sigma_0, f^{(m-1)}) + \int_{\sigma_0}^{\sigma}\lambda_{\nu}(t, f^{(m-1)})\,dt.$$

Differentiating (3.4) and (3.5), we get

(3.6) $$\frac{\mu^{(1)}(\sigma, f^{(m)})}{\mu(\sigma, f^{(m)})} = \lambda_{\nu}(\sigma, f^{(m)}).$$

(3.7) $$\frac{\mu^{(1)}(\sigma, f^{(m-1)})}{\mu(\sigma, f^{(m-1)})} = \lambda_{\nu}(\sigma, f^{(m-1)}),$$

for almost all values of $\sigma > \sigma_0 \geq 0$.

Therefore, on using the result of (3.3), we can deduce from (3.6) and (3.7)

(3.8) $$\frac{\mu^{(1)}(\sigma, f^{(m)})}{\mu^{(1)}(\sigma, f^{(m-1)})} \sim \frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f^{(m-1)})},$$

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* This result has also been obtained by [3].

** This result has been obtained by [3] and also by [4] with the conditions $\lambda \geq \delta > 0$ and

$$\lim_{\sigma\to\infty}\frac{\log\lambda_{\nu}(\sigma, f^{(m)}) - \log\lambda_{\nu}(\sigma, f^{(m-1)})}{\sigma} = 0, \quad (m = 1, 2, \ldots).$$

for large σ. Now, R. P. SRIVASTAV [2] has obtained

$$\liminf_{\sigma\to\infty}\left\{\sigma^{-1}\log\left(\frac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)}\right)\right\}=\lambda.$$

Therefore, for large σ.

$$\frac{\mu(\sigma, f^{(1)})}{\mu(\sigma, f)}>e^{\sigma(\lambda-\varepsilon)}\geq 1, \text{ since } \lambda>0.$$

Similarly, for $m=1, 2, \ldots\ldots$, we can prove

$$\frac{\mu(\sigma, f^{(m)})}{\mu(a, f^{(m-1)})}>1.$$

So from (3.8), we get

$$\mu^{(1)}(\sigma, f^{(m)}>\mu^{(1)}(o, f^{(m-1)}), \text{ for } \sigma>\sigma_0\geq 0.$$

Putting $m=1, 2, \ldots\ldots$, we get

$$\mu^{(1)}(o, f)<\mu^{(1)}(a, f^{(1)})<\mu^{(1)}(o, f^{(2)})<\ldots\ldots .$$

4. Theorem 2. *Let*

$$f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}$$

be an integral function of linear order $\varrho\,(0<\varrho<\infty)$, type T and lower type t. Then, if $o\to\infty$ through values outside a set of measure zero and $m=1, 2, \ldots\ldots s$,

$$\limsup_{\sigma\to\infty}\left\{\sigma\, e^{-\varrho\sigma}\left(\frac{\mu(\sigma, f^{(m)}}{M(\sigma, f)}\right)^{1/m}\right\}\leq T\leq\limsup_{\sigma\to\infty}\left\{\sigma\, e^{-\varrho\sigma}\left(\frac{M(\sigma, f^{(m)}}{M(\sigma, f)}\right)^{1/m}\right\}, \tag{4.1}$$

provided the linear lower order of $f(s)$ is greater than zero.

$$\limsup_{\sigma\to\infty}\left\{\sigma\, e^{-\varrho\sigma}\left(\frac{\mu(\sigma, f^{(m)}}{M(\sigma, f)}\right)^{1/m}\right\}\leq t\leq\liminf_{\sigma\to\infty}\left\{\sigma\, e^{-\varrho\sigma}\left(\frac{M(\sigma, f^{(m)})}{M(\sigma, f)}\right)^{1/m}\right\}, \tag{4.2}$$

where $\mu(\sigma, f^{(m)}$ and $M(o, f^{(m)})$ are ihe maximum term and least upper bound of the m^{th} derivative of $f(s)$ respectively.

Proof of Theorem 2. S. N. SRIVASTAV [1] has obtained the following inequality

$$M(\sigma, f^{(m)}\geq\frac{M(\sigma, f^{(m-1)})\log M(\sigma, f^{(m-1)})}{\sigma}, \text{ for } \sigma>\sigma_m. \tag{4.3}$$

Further, he has proved that the sequence

$$M(\sigma, f),\ M(\sigma, f^{(1)}),\ M(\sigma, f^{(2)}),\ \ldots\ldots,\ M(\sigma, f^{(s)}),$$

where $\sigma > \sigma_0$ and $\sigma_0 = \max. \ (\sigma_1, \sigma_2, \ldots\ldots, \sigma_s)$, forms an increasing sequence provided $f(s)$ is an integral function of linear lower order λ, $\lambda > 0$.

From (4.3), we have, for $\sigma > \sigma_0$ where $\sigma_0 = \max. \ (\sigma_1, \sigma_2, \ldots, \sigma_m)$

$$\frac{M(\sigma, f^{(m)})}{M(\sigma, f)} \geq \frac{\log M(\sigma, f) \cdot \log M(\sigma, f^{(1)}) \ldots\ldots \log M(\sigma, f^{(m-1)})}{\sigma^m} \tag{4.4}$$

and if $\lambda > 0$

$$\frac{M(\sigma, f^{(m)})}{M(\sigma, f)} \geq \left\{\frac{\log M(\sigma, f)}{\sigma}\right\}^m. \tag{4.5}$$

Therefore, on taking limits of both the sides in (4.4) and (4.5) respectively, we get

$$\liminf_{\sigma\to\infty} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{M(\sigma, f^{(m)})}{M(\sigma, f)}\right)^{1/m}\right\} \geq t \tag{4.6}$$

and

$$\limsup_{\sigma\to\infty} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{M(\sigma, f^{(m)})}{M(\sigma, f)}\right)^{1/m}\right\} \geq T, \text{ for } \lambda > 0. \tag{4.7}$$

Now, from (2.4), we have

$$\frac{\mu(\sigma, f^{(p)})}{\mu(\sigma, f^{(p-1)})} \leq \lambda_{\nu(\sigma, f^{(p)})}\, .$$

Therefore, writing the above inequality for $p = 1, 2, \ldots\ldots, m$ and multiplying together, we get

$$\frac{\mu(\sigma, f^{(m)})}{\mu(\sigma, f)} \leq \lambda_{\nu(\sigma, f^{(1)})} \ldots\ldots \lambda_{\nu(\sigma, f^{(m)})}$$

or,

$$\left\{\frac{\mu(\sigma, f^{(m)})}{M(\sigma, f)}\right\}^{1/m} \leq \lambda_{\nu(\sigma, f^{(m)})}\, .$$

Hence on using Lemma 1 and Lemma 2, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\sigma\, e^{-\varrho\sigma} \left(\frac{\mu(\sigma, f^{(m)})}{M(\sigma, f)}\right)^{1/m}\right\} \leq \begin{matrix}T\\ t\end{matrix}\,, \tag{4.8}$$

where $\sigma \to \infty$ through values outside a set of measure zero. The results (4.1) and (4.2) follow from (4.6), (4.7) and (4.8).

5. Theorem 3. *Let*

$$f(s) = \sum_{n=1}^{\infty} a_n\, e^{s\lambda_n}$$

be an integral function of linear order $\varrho\,(0<\varrho<\infty)$ *type* T *and lower type* t, *then*

$$(5.1)\qquad \liminf_{\sigma\to\infty}\left\{e^{-\sigma\lambda_\nu(\sigma,f)}\int_{\sigma_0}^{\sigma} e^{t(\varrho+\lambda_\nu(t,f))}\,dt\right\}\leq 1/\varrho\,T\leq 1/\varrho\,t$$

$$\leq \limsup_{\sigma\to\infty}\left\{e^{-\sigma\lambda_\nu(\sigma,f)}\int_{\sigma_0}^{\sigma} e^{t(\varrho+\lambda_\nu(t,f))}\,dt\right\},$$

where $\sigma\to\infty$ *through values outside a set of measure zero.*

We first prove the following two Lemmas:

Lemma 3. If $K(\sigma)$ is a positive, real function of σ, continuous almost everywhere in (σ_0, ∞) and

$$\limsup_{\sigma\to\infty}\left(e^{-\varrho\sigma}\log K(\sigma)\right)=\alpha,$$

then

$$(5.2)\qquad \liminf_{\sigma\to\infty}\left\{\left(K(\sigma)\right)^{-1}\int_{\sigma_0}^{\sigma} K(t)\,e^{\varrho t}\,di\right\}\leq 1/\varrho\alpha.$$

Proof: Let

$$(5.3)\qquad \Phi(\sigma)=\int_{\sigma_0}^{\sigma} K(t)\,e^{\varrho t}\,dt.$$

The inequality (5.2) is obviously true if $\alpha=0$. Hence we consider the case $\alpha>0$. Suppose first $\alpha>0$ and finite and suppose (5.2) does not hold.

Then, we have, for $\sigma\geq\delta=\delta(x)>\sigma_0$

$$\Phi(\sigma)>x\,K(\sigma),\quad \text{where}\quad x>1/\varrho\alpha.$$

From (5.3) it follows that $\Phi'(\sigma)$ exists and

$$\Phi'(\sigma)=K(\sigma)\,e^{\varrho\sigma},$$

almost everywhere. Therefore,

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)}<\frac{e^{\varrho\sigma}}{x},$$

almost everywhere. Thus for large σ

$$\log\Phi(\sigma)=\log\Phi(\sigma_0)+\int_{\sigma_0}^{\sigma}\frac{\Phi'(t)}{\Phi(t)}\,dt<\log\Phi(\sigma_0)+1/\varrho\,x\,(e^{\varrho\sigma}-\varrho^{\sigma_0}).$$

Hence

$$\log x + \log K(\sigma) < \frac{1}{\varrho x} e^{\varrho\sigma} + o(1)$$

or,

$$\limsup_{\sigma\to\infty} \left(e^{-\varrho\sigma} \log K(\sigma)\right) \leq 1/\varrho x < \alpha.$$

which contradicts the hypothesis. Hence the Lemma is proved for $0 \leq \alpha < \infty$. If $\alpha = \infty$, then we take x to he arbitrary small.

Lemma 4. If $K(\sigma)$ satisfies the conditions of Lemma 3 and

$$\liminf_{\sigma\to\infty} \left(e^{-\varrho\sigma} \log K(\sigma)\right) = \beta,$$

then

$$\limsup_{\sigma\to\infty} \left\{\left(K(\sigma)\right)^{-1} \int_{\sigma_0}^{\sigma} K(t)\, e^{\varrho t}\, dt\right\} \geq 1/\varrho \beta.$$

This can easily he proved if we adopt the method of proof of Lemma 3.

Proof of Theorem 3. If we take $K(\sigma) = e^{\lambda_\nu(\sigma, f)}$ in the above two Lemmas and combine the inequalities thus obtained, we get the required result.

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ÖZET

Bu araştırmanın gayesi bütün düzlemde bir DIRICHLET seri açılımı ile gösterilen bir tam fonksiyonun maksimum teriminin türevlerinin bazı özelliklerini incelemektir. Maksimum modül ve maksimum terimin türevinin bir ifadesi de ayrıca elde edilmiş bulunmaktadır.