ON CONVEX FUNCTIONS AND THEIR APPLICATIONS TO ENTIRE FUNCTIONS

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In this paper the growth of an indefinitely increasing function of a real variable has been studied in relation to another function by introducing the notion of order and type. An attempt has been made here to unify «certain aspects» of the two theories of entire functions defined by Taylor series and Dirichlet series respectively, which have so far been treated separately by different workers in the two fields. Some applications given in conclusion are intended to emphasize this fact.

1. Let $F(x)$ and $\psi(x)$ be two indefinitely increasing functions of the real variable x defined by

$$\log F(x) = \log F(x_0) + \int_{x_0}^{x} \alpha(t)\, g(t)\, dt \tag{1.1}$$

and

$$\psi(x) = \psi(x_0) + \int_{x_0}^{x} g(t)\, dt \tag{1.2}$$

where $\alpha(t)$ and $g(t)$ are both positive for $0 \leq x_0 \leq t \leq x$ and integrable in the sense of Lebesgue. We say that $\psi(x)$ belongs to the class A or to B according as for any positive constant k

$$\psi(kx) - \psi(x) = O(1) \tag{1.3A}$$

$$\psi(x+k) - \psi(x) = O(1) \tag{1.3B}$$

as x approaches infinity. Since $\psi(x)$ increases indefinitely with x, we have $\log \psi(x) = o\{\psi(x)\}$ and from (1.3A), it follows that $\psi(kx) \sim \psi(x)$ while (1.3B) gives $\psi(x+k) \sim \psi(x)$. Since $\alpha(t)$ and $g(t)$ are both positive, so are the functions $\log F(x)$ and $\psi(x)$. Further, if we take $\alpha(x)$ to be also nondecreasing, then it follows easily that *log $F(x)$ is a convex function of $\psi(x)$*.

In this paper we study the growth of the functions $F(x)$ and $\alpha(x)$ in relation to the function $\psi(x)$ and obtain a number of relations. It will be seen that many known results of the theory of entire functions representable by Taylor series and those defined by Dirichlet's series, become the direct consequences of the results obtained here.

Throughout in our discussions we shall assume that $F(x)$ and $\psi(x)$, as defined in (1.1) and (1.2), are both indefinitely increasing functions which are also positive since $\alpha(t)$ and $g(t)$ are both positive for $0 \leq x_0 \leq t < x$ and that $\alpha(t)$ is also nondecreasing, thus making $\log F(x)$ a convex function of $\psi(x)$ which will be taken to belong either to the class A or to the class B as defined above.

2. Theorem 1.

If

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log F(x)}{\psi(x)} = \begin{matrix}\varrho\\ \lambda\end{matrix} \tag{2.1}$$

then

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \alpha(x)}{\psi(x)} = \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{2.2}$$

Proof:

First suppose that $\psi(x)$ belongs to the class A; then for $k > 1$, we have

$$\log F(kx) = \log F(x_0) + \int_{x_0}^{x} \alpha(t)\, g(t)\, dt + \int_{x}^{kx} \alpha(t)\, g(t)\, dt > \int_{x}^{kx} \alpha(t)\, g(t)\, dt$$

$$\geq \alpha(x)\,[\psi(kx) - \psi(x)].$$

As $\psi(x)$ satisfies the condition (1.3A), $\psi(kx) \sim \psi(x)$ and hence we get,

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \alpha(x)}{\psi(x)} \leq \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log F(x)}{\psi(x)}. \tag{2.3}$$

Also,

$$\log F(x) = \log F(x_0) + \int_{x_0}^{x} \alpha(t)\, g(t)\, dt \leq \log F(x_0) + \alpha(x)\,[\psi(x) - \psi(x_0)].$$

Hence,

$$\frac{\log \log F(x)}{\psi(x)} \leq \frac{\log \alpha(x)}{\psi(x)} + o(1).$$

Thus

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \alpha(x)}{\psi(x)} \geq \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log F(x)}{\psi(x)}. \tag{2.4}$$

Now, (2.2) follows from (2.1), (2.3) and (2.4).

Next suppose that $\psi(x)$ belongs to the class B; then

$$\log F(x+k) = \log F(x_0) + \int_{x_0}^{x+k} \alpha(t).\ g(t)\ dt > \int_{x}^{x+k} \alpha(t).\ g(t)\ dt$$

$$\geq \alpha(x) [\, \psi(x+k) - \psi(x) \,].$$

As $\psi(x+k) - \psi(x) = O(1)$, we get $\psi(x+k) \sim \psi(x)$ and hence

$$\text{(2.5)} \qquad \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \alpha(x)}{\varphi(x)} \leq \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log F(x)}{\psi(x)} .$$

Hence, (2.2) follows from (2.1), (2.4) and (2.5).

Corollary:

If $F'(x)$ be the derivative of $F(x)$ [which exists for almost all values of $x \geq x_0$], then

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \dfrac{F'(x)}{F(x) \,.\, g(x)}}{\psi(x)} = \begin{matrix}\varrho\\ \lambda\end{matrix} .$$

Since from (1.1), we have on differentiation $\dfrac{F'(x)}{F(x)} = \alpha(x)\ .\ g(x)$ for almost all values of $x \geq x_0$, the corollary follows from (2.2) on substituting $\dfrac{F'(x)}{F(x)\,.\,g(x)}$ for $\alpha(x)$.

Henceforth, we shall refer to the constants ϱ and λ as defined in (2.1) by «ψ—order» and «lower ψ—order» respectively of the functions $F(x)$ which will he said to be of «regular ψ—growth» when $\varrho = \lambda$. The justification for this lies in the fact that ϱ and λ depend on the function $\psi(x)$.

Theorem 2.

Let ϱ and λ be respectively the ψ—order and the ψ—lower order of the function $F(x)$; if

$$\text{(2.6)} \qquad \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\alpha(x)}{\exp.[\,\varrho\,.\,\psi(x)\,]} = \begin{matrix}\gamma\\ \delta\end{matrix} \qquad (0 < \varrho < \infty)$$

then

$$\text{(2.7)} \qquad \frac{\delta}{\varrho\gamma} \leq \lim_{x\to\infty} \inf \frac{\log F(x)}{\psi(x)} \leq \frac{1}{\varrho} \leq \frac{1}{\lambda} \leq \lim_{x\to\infty} \sup \frac{\log F(x)}{\alpha(x)} \leq \frac{\gamma}{\varrho\delta} .$$

Proof:

From (1.1) we have,

$$\frac{F'(x)}{F(x)} = \alpha(x) \cdot g(x) \tag{2.8}$$

everywhere except for a set of measure zero. Let

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \text{iuf}\end{matrix} \frac{\log F(x)}{\alpha(x)} = \frac{c}{d},$$

therefore for $\varepsilon > 0$, we have,

$$d - \varepsilon < \frac{\log F(x)}{\alpha(x)} < c + \varepsilon.$$

hence in view of (2.8), we have,

$$(d-\varepsilon)\frac{F'(x)}{F(x) \cdot \log F(x)} < g(x) < (c+\varepsilon)\frac{F'(x)}{F(x) \cdot \log F(x)}$$

or

$$(d-\varepsilon)\int_{x_0}^{x} \frac{F'(t)}{F(t) \cdot \log F(t)}\, dt < \int_{x_0}^{x} g(t)\, dt < (c+\varepsilon)\int_{x_0}^{x} \frac{F'(t)}{F(t) \cdot \log F(t)}\, dt$$

or

$$O(1) + (d-\varepsilon) \log\log F(x) < [\,\psi(x) - \psi(x_0)\,] < (c+\varepsilon)\log\log F(x) + O(1)$$

or

$$\left[\frac{(d-\varepsilon)\log\log F(x)}{\psi(x)} + o(1)\right] < [\,1 - o(1)\,] < \left[\frac{(c+\varepsilon)\log\log F(x)}{\psi(x)} + o(1)\right].$$

Hence

$$\lim_{x\to\infty} \sup \frac{\log\log F(x)}{\psi(x)} \leq \frac{1}{d}$$

and

$$\lim_{x\to\infty} \inf \frac{\log\log F(x)}{\psi(x)} \geq \frac{1}{c}$$

or

$$d \leq \frac{1}{\varrho} \qquad \text{and} \qquad c \geq \frac{1}{\lambda}.$$

Hence

$$\lim_{x\to\infty} \inf \frac{\log F(x)}{\alpha(x)} \leq \frac{1}{\varrho} \leq \frac{1}{\lambda} \leq \lim_{x\to\infty} \sup \frac{\log F(x)}{\alpha(x)}. \tag{2.9}$$

Again, from (2.6), we have,

$$\delta - \varepsilon < \frac{\alpha(x)}{e^{[\varrho \cdot \psi(x)]}} < \gamma + \varepsilon$$

hence in view of (2.8), we have

$$(\delta - \varepsilon) \cdot g(x) \cdot e^{\varrho \cdot \psi(x)} < \frac{F'(x)}{F(x)} < (\gamma + \varepsilon) \cdot g(x) \cdot e^{\varrho \cdot \psi(x)}.$$

Integrating the above inequalities between the limits x_0 to x and then dividing by $\alpha(x)$ we get

$$\left[(\delta - \varepsilon) \cdot \frac{1}{\varrho} \cdot \frac{e^{\varrho \cdot \psi(x)}}{\alpha(x)} + o(1)\right] < \frac{\log F(x)}{\alpha(x)} < \left[(\gamma + \varepsilon) \cdot \frac{1}{\varrho} \cdot \frac{e^{\varrho \cdot \psi(x)}}{\alpha(x)} + o(1)\right].$$

Therefore,

$$\frac{\delta}{\varrho \cdot \gamma} \leq \lim_{x \to \infty} \inf \frac{\log F(x)}{\alpha(x)} \leq \lim_{x \to \infty} \sup \frac{\log F(x)}{\alpha(x)} \leq \frac{\gamma}{\varrho \cdot \delta}. \tag{2.10}$$

From (2.9) and (2.10), the result of (2.7) follows.

Corollary :

If $\gamma = \delta$, the function $F(x)$ is of regular ψ—growth.

Theorem 3.

If $0 < x_1 < x_2$

then

$$\alpha(x_1) \leq \frac{\log F(x_2) - \log F(x_1)}{\psi(x_2) - \psi(x_1)} \leq \alpha(x_2). \tag{2.11}$$

Proof :

From (1.1) and (1.2) we have

$$\log F(x_2) = \log F(x_1) + \int_{x_1}^{x_2} \alpha(t) \cdot g(t)\, dt$$

and

$$\psi(x_2) = \psi(x_1) + \int_{x_1}^{x_2} g(t) \cdot dt.$$

Hence

$$\log F(x_2) - \log F(x_1) \leq \alpha(x_2)\,[\,\psi(x_2) - \psi(x_1)\,]. \tag{2.12}$$

Also

(2.13) $$\log F(x_2) - \log F(x_1) \geq \alpha(x_1)\ [\psi(x_2) - \psi(x_1)].$$

From (2.12) and (2.13), the result in (2.11) follows.

Theorem 4.

Let $F(x)$ and $\psi(x)$ be positive and indefinitely increasing functions of the real variable x defined by (1.1) *and* (1.2). *Farther let $\varphi(x)$ be defined by*

(2.14) $$\log \varphi(x) = \log \varphi(x_0) + \int_{x_0}^{x} \beta(t)\ g(t)\ dt$$

where $\beta(x) \geq \alpha(x)$; then $\varphi(x)$ is also a positive and indefinitely increasing function. Farther if,

(2.15) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \dfrac{\varphi(x)}{F(x)}}{\psi(x)} = \begin{matrix}A\\ B\end{matrix}$$

and

(2.16) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\ [\beta(x) - \alpha(x)] = \begin{matrix}P\\ Q\end{matrix}$$

then

(2.17) $$Q \leq B \leq A \leq P.$$

Proof:

That $\varphi(x)$ is also positive and indefinitely increasing, is evident from the definition of $F(x)$ and $\varphi(x)$, and the fact that $\beta(x) \geq \alpha(x)$.

From (1.1) and (2.14), we get

$$\log \frac{\varphi(x)}{\varphi(x_0)} - \log \frac{F(x)}{F(x_0)} = \int_{x_0}^{x} [\beta(t) - \alpha(t)].\ g(t)\, dt.$$

Therefore

$$\frac{1}{\psi(x)}\left[\log \frac{\varphi(x)}{F(x)} + O(1)\right] = \frac{1}{\psi(x)}\left[\int_{x_0}^{x} \{\beta(t) - \alpha(t)\}\,.\, g(t)\ dt\right].$$

But, from (2.16) we have for any $\varepsilon > 0$ and sufficiently large x

$$Q - \varepsilon < [\beta(x) - \alpha(x)] < P + \varepsilon.$$

Hence

$$\frac{Q-\varepsilon}{\psi(x)}\int_{x_0}^{x} g(t)\,dt < \frac{1}{\psi(x)}\left[\log\frac{\varphi(x)}{F(x)}+O(1)\right] < \frac{P+\varepsilon}{\psi(x)}\int_{x_0}^{x} g(t)\,.\,dt.$$

On proceeding to limits the result of (2.17) follows, since,

$$\int_{x_0}^{x} g(t)\,dt = \psi(x)-\psi(x_0) \quad \text{and} \quad \psi(x)\to\infty\,.$$

Corollary :

If the limit in (2.16) *exists, so does the limit in* (2.15) *and then the two limits are equal.*

Remark.

If $P<\infty$, then $\alpha(x)\sim\beta(x)$ and therefore in view of theorem 1, it follows that both the functions $F(x)$ and $\varphi(x)$ have the same ψ—order and the same lower ψ—order.

3. A better estimate of the growth of the function $F(x)$ in relation to the function $\psi(x)$ is obtained if we consider the limit of $\dfrac{\log F(x)}{\exp.[\varrho.\psi(x)]}$. Thus, let

(3.1) $$\lim_{x\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log F(x)}{e^{\varrho.\psi(x)}} = \begin{matrix}T\\ t\end{matrix}\,,\quad (0\leq t\leq T\leq\infty).$$

where $\varrho\,(0<\varrho<\infty)$ is the ψ—order of $F(x)$. We define T to be the ψ—type and t the lower ψ—type of the function $F(x)$ of ψ—order ϱ $(0<\varrho<\infty)$ and in case, the limit in (3.1) exists, *i.e.*, $T=t$, we say that $F(x)$ is of «perfectly ψ—regular growth».

Lemma :

If $F(x)$ is of «perfectly ψ—regular growth» of ψ—type T $(0<T<\infty)$ of ψ—order ϱ $(0<\varrho<\infty)$, then it is necessarily of ψ—regular growth.

Proof :

We have, since $F(x)$ is of perfectly ψ—regular growth,

$$\log F(x)\sim T\,.\,e^{\varrho.\psi(x)}\,,$$

hence,

$$\log\log F(x)\sim\log T+\varrho\,.\,\psi(x).$$

Dividing both sides by $\psi(x)$ and proceeding to limits we get, since $0<T<\infty$ and $\psi(x)\to\infty$,

$$\lim_{x\to\infty} \frac{\log \log F(x)}{\psi(x)} = \varrho.$$

Theorem 5.

Let ϱ $(0 < \varrho < \infty)$, be the ψ—order of the function $F(x)$; if

$$\lim_{x\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} = \begin{matrix}T \\ t\end{matrix} \qquad (0 \leq t \leq T \leq \infty)$$

and

$$\lim_{x\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(x)}{e^{\varrho \cdot \psi(x)}} = \begin{matrix}\gamma \\ \delta\end{matrix}.$$

then

$$\delta \leq \varrho\, t \leq \varrho\, T \leq \gamma \tag{3.2}$$

Proof:

We have for any $\varepsilon > 0$ and $x > x_0$

$$(\gamma + \varepsilon)\; e^{\varrho \cdot \psi(x)} > \alpha(x) > (\delta - \varepsilon)\; e^{\varrho \cdot \psi(x)}. \tag{3.3}$$

Therefore, from (1.1), we get,

$$\log F(x) > \log F(x_0) + (\delta - \varepsilon) \int_{x_0}^{x} e^{\varrho \cdot \psi(x)} \cdot g(t)\; dt = \log F(x_0) + \frac{\delta - \varepsilon}{\varrho} [e^{\varrho \cdot \psi(x)} - e^{\varrho \cdot \psi(x_0)}].$$

Hence,

$$\frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} > \frac{\delta - \varepsilon}{\varrho} + o(1).$$

Thus

$$\lim_{x\leftarrow\infty} \inf \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} \geq \frac{\delta}{\varrho}. \tag{3.4}$$

Next, taking the first inequality in (3.3) and using it in (1.1), we similarily obtain,

$$\lim_{x\to\infty} \sup \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} \leq \frac{\gamma}{\varrho} \tag{3.5}$$

and hence the result of (3.2) follows.

Corollary:

If $\gamma = \delta$, then $F(x)$ is of perfectly regular ψ—growth.

APPLICATIONS

4. Here we give some applications of the results derived in the previous sections to entire functions.

First we consider the case of TAYLOR series. Let $f(z) = \sum_{0}^{\infty} a_n z^n$, $z = x + iy$. a_n real or complex, be an entire function of order ϱ and lower order λ; $M(r)$ denote its maximum modulus, $\mu(r)$ its maximum term of rank $\nu(r)$ for $|z| = r$.

It is known [1, 31; 2, 10] that

$$\log \mu(\gamma) = \log \mu(\gamma_0) + \int_{\gamma_0}^{\gamma} \frac{\nu(t)}{t}\, dt \tag{4.1}$$

and

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(r)}{\log r} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu(r)}{\log r} = \frac{\varrho}{\lambda}. \tag{4.2}$$

Since $\nu(t)$ is positive and non-decreasing and

$$\log r = \log r_0 + \int_{\gamma_0}^{\gamma} \frac{1}{t}\, dt,$$

we observe that $\mu(r)$ has a representation similar to that of $F(x)$ with

$$\alpha(t) = \nu(t) \text{ and } g(t) = 1/t,$$

consequently, the result of the second equality in (4.2) follows from theorem 1.

Also, since [1, 27]

$$\log M(r) = \log M(r_0) + \int_{\gamma_0}^{\gamma} \frac{W(t)}{t}\, dt$$

where $W(t)$ is a positive and indefinitely increasing function, we get from corollary to theorem 1,

$$\lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \dfrac{r.\, M'(r)}{M(r)}}{\log r} = \frac{\varrho}{\lambda} \tag{4.3}$$

where $M'(r)$ is the derivative of $M(r)$ which exists for almost all values of r [14, 34].

Similarily, from theorem 2, we get the result [4, 10]; [8, 80-82]

$$(4.4)\qquad \frac{\delta}{\varrho\gamma} \leq \lim_{r\to\infty} \inf \frac{\log \mu(\gamma)}{\nu(\gamma)} \leq \frac{1}{\varrho} \leq \frac{1}{\lambda} \leq \lim_{\gamma\to\infty} \sup \frac{\log \mu(\gamma)}{\nu(r)} \leq \frac{\gamma}{\varrho\delta}$$

where

$$\lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\nu(\gamma)}{\gamma^{\varrho}} = \frac{\gamma}{\delta}.$$

From (2.11) of theorem 3, we get the inequalities [5, 53]

$$(4.5)\qquad \left[\frac{r_2}{r_1}\right]^{\nu(\gamma_1)} \leq \frac{\mu(\gamma_2)}{\mu(\gamma_1)} \leq \left(\frac{r_2}{r_1}\right)^{\nu(\gamma_2)}$$

where $0 < r_1 \leq r_2$.

Again, if T, t be respectively the type and lower type of $f(z)$ of order $\varrho\,(0 < \varrho < \infty)$, then the theorem 5 yields the inequalities [15, 220]

$$(4.6)\qquad \delta \leq \varrho\, t \leq \varrho\, T \leq \gamma.$$

The mean value of $f(z)$ is defined as

$$M_q(\gamma) = \frac{1}{2\pi}\left[\int_0^{2\pi} |\, f(\gamma \,.\, e^{i\vartheta})\,|\, d\vartheta\right]^{\frac{1}{q}}, \quad q > 0.$$

It is known [6, 748] that

$$\log M_q(r) = \log M_q(r_0) + \int_{\gamma_0}^{\gamma} \frac{m_q(t)}{t}\, dt.$$

Here $\log M_q(r)$ is an indefinitely increasing function being a convex function of $\log r$. Therefore on using theorem 1, we get [7, 193]

$$(4.7)\qquad \lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M_q(r)}{\log r} = \frac{\varrho}{\lambda} = \lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_q(r)}{\log r}.$$

If

$$\lim_{\gamma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{m_q(r)}{r^{\varrho}} = \frac{c}{d}$$

we get from (2.7)

$$(4.8)\qquad \frac{d}{\varrho \,.\, c \,.\, q} \leq \lim_{\gamma\to\infty} \inf \frac{\log M_q(\gamma)}{m_q(\gamma)} \leq \frac{1}{\varrho \,.\, q} \leq \frac{1}{\lambda \,.\, q} \leq \lim_{\gamma\to\infty} \sup \frac{\log M_q(\gamma)}{m_q(\gamma)} \leq \frac{c}{\varrho \,.\, d \,.\, q}.$$

Now consider the entire function

$$f(s) = \sum_{n=1}^{\infty} a_n\, e^{\lambda_n . s}$$

where $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$, $\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0$

defined by a DIRICHLET series. Let, as usual,

$$M(\sigma) = \underset{-\infty \leq t \leq \infty}{\text{l. u. b.}} \; |f(\sigma + it)|$$

and $\mu(\sigma)$ denotes the maximum term of rank $N(\sigma)$ for $Re(s) = \sigma$. The RITT-order [8, 78] ϱ and lower order λ are given by

$$(4.9) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Also, it is known [9, 67] that

$$(4.10) \qquad \log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_N(t)\, dt.$$

Since $\lambda_{N(\sigma)}$ is a non decreasing function and

$$\sigma = \sigma_0 + \int_{\sigma_0}^{\sigma} dt,$$

we again have analogue between the functions $\mu(\sigma)$ and $F(x)$ with $\psi(\sigma) = \sigma$ which belongs to class B. Consequently, many results which have been obtained separately for the case of entire functions defined by DIRICHLET series follow directly from the theorems proved here. We list below a few of them.

(*i*) From theorem 1, we get, [9, 69-73]

$$(4.11) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M(\sigma)}{\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log \mu(\sigma)}{\sigma}$$
$$= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \lambda_{N(\sigma)}}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

(*ii*) From Corollary to theorem 1, we have [10, lemma 2]

$$(4.12) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \dfrac{M'(\sigma)}{M(\sigma)}}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}$$

where $M'(\sigma)$ denotes the derivative of $M(\sigma)$.

(*iii*) If

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{N(\sigma)}}{e^{\varrho\sigma}} = \begin{matrix}\gamma\\ \delta\end{matrix}$$

then from theorem 2, we obtain [11, 135]

(4.13) $$\frac{\delta}{\varrho \cdot \gamma} \leq \lim_{\sigma\to\infty} \inf \frac{\log \mu(\sigma)}{\lambda_{N(\sigma)}} \leq \frac{1}{\varrho} \leq \frac{1}{\lambda} \leq \lim_{\sigma\to\infty} \sup \frac{\log \mu(\sigma)}{\lambda_{N(\sigma)}} \leq \frac{\gamma}{\varrho \cdot \delta}.$$

(*iv*) From (2.11) of theorem 3 we get the inequalities [11, 139]

(4.14) $$\left[\frac{e^{\sigma_2}}{e^{\sigma_1}}\right]^{\lambda_{N(\sigma_1)}} \leq \frac{\mu(\sigma_2)}{\mu(\sigma_1)} \leq \left[\frac{e^{\sigma_2}}{e^{\sigma_1}}\right]^{\lambda_{N(\sigma_2)}}$$

if $$0 < \sigma_1 \leq \sigma_2.$$

(*v*) If T and t be respectively the type and lower type of $f(s)$ of order $\varrho\,(0 < \varrho < \infty)$, theorem 5 yields the inequalities [11, 141]

(4.15) $$\delta \leq \varrho\, t \leq \varrho\, T \leq \gamma.$$

(*vi*) It is known [12, 707] that if $f(s)$ is of linearly regular growth and

$$\operatorname*{limit}_{\sigma\to\infty} \left[\lambda_{N(\sigma, f^{(r)})} - \lambda_{N(\sigma, f)}\right]$$

exists then $f(s)$ is of finite order ϱ such that

(4.16) $$\lim_{\sigma\to\infty} \left[\lambda_{N(\sigma, f^{(r)})} - \lambda_{N(\sigma, f)}\right] = n\,\varrho$$

for $n = 1, 2, \ldots$, $\lambda_{N(\sigma, f)}$ and $\lambda_{N(\sigma, f^{(n)})}$ being the ranks of the maximum terms in $f(s)$ and its n^{th} derivative $f^{(n)}(s)$, respectively.

Since,

(4.17) $$\log \mu(\sigma, f) = \log \mu(\sigma_0, f) + \int_{\sigma_0}^{\sigma} \lambda_{N(t, f)}\, dt,$$

(4.18) $$\log \mu(\sigma, f^{(n)}) = \log \mu(\sigma_0, f^{(n)}) + \int_{\sigma_0}^{\sigma} \lambda_{N(t, f^{(n)})}\, dt$$

and also [13, 89]

(4.19) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \dfrac{\mu(\sigma, f^{(n)})}{\mu(\sigma, f)}}{\sigma} = \begin{matrix} n\varrho \\ n\lambda \end{matrix}.$$

Hence applying the result of theorem 4 and its corollary we obtain the result in (4.16) (*).

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ÖZET

Sıra ve tip kavramları tarif edilmek suretiyle, reel bir değişkenin sınırsız artan bir fonksiyonunun artışı, diğer bir fonksiyona göre incelenmektedir. Bir taraftan Taylor serileri vasıtasiyle, diğer taraftan Dirichlet serileri yolu ile tarif edilebilen tam fonksiyonların bu iki ayrı koldaki teorilerinin değişik yazarlar tarafından ayrı tutularak incelenen bazı noktaları bu araştırmada birleştirilmeye çalışılmıştır. Elde edilen sonuçlar birçok problemlere uygulanarak bu husus tebarüz ettirilmiştir.