THE PERIHELION MOTION AND DEFLECTION OF LIGHT

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The object of this paper is to find the perihelion motion and deflection of light by the approximation method.

1. Introduction. EDDINGTON [1, 88] solved the equations connected with the motion of the one body problem, to predict the perihelion motion and deflection of light. INFELD & PLEBANSKI [2, 144] derived the same result for the perihelion motion by the E.I.H. approximation method.

In the case of small velocities we have shown that the above results can be obtained by using a particular solution of EINSTEIN's gravitational equations $G_{\mu\nu}=0$ for an isolated particle. The deduction is extremly simple.

2. Perihelion Motion. The particular solution of EINSTEIN's gravitational equations $G_{\mu\nu}=0$ in the isotropic coordinates is

$$(2.1)\qquad ds^2=-\left(1+\frac{m}{2r_1}\right)^4(dr_1^2+r_1^2\,d\vartheta^2+r_1^2\sin^2\vartheta d\varphi^2)+\frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2}\,dt^2.$$

Putting $r_1=-r$ in (2.1), we have

$$(2.2)\qquad ds^2=-\left(1-\frac{m}{2r}\right)^4(dr^2+r^2d\vartheta^2+r^2\sin^2\vartheta d\varphi^2)+\frac{\left(1+\frac{m}{2r}\right)^2}{\left(1-\frac{m}{2r}\right)^2}\,dt.$$

Choosing coordinates such that the planet moves in the plane $\varphi=\pi/2$, from (2.2), we have

$$(2.3)\qquad \left(\frac{ds}{dt}\right)^2=-\left(1-\frac{m}{2r}\right)^4\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\varphi}{dt}\right)^2\right]+\frac{\left(1+\frac{m}{2r}\right)^2}{\left(1-\frac{m}{2r}\right)^2}\,.$$

Following TOLMAN [3, 208], we have

(2.4) $$\frac{ds}{dt} = 1.$$

On using (2.4), (2.3) yields

(2.5) $$\left(1 - \frac{m}{2r}\right)^4 \left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\varphi}{dt}\right)^2\right] = \frac{\frac{2m}{r}}{\left(1 - \frac{m}{2r}\right)^2}.$$

Neglecting the cubes and higher powers of the small term m/r, we get

(2.6) $$\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\varphi}{dt}\right)^2 = \frac{2m}{r} + \frac{6m^2}{r^2}.$$

Using the Law of Conservation of Momenta, which in the polar coordinates system (r, φ) is

(2.7) $$r^2 \frac{d\varphi}{dt} = h,$$

and putting

(2.8) $$u = \frac{1}{r}$$

in (2.6), we have

(2.9) $$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{2mu}{h^2} + \frac{6m^2u^2}{h^2}.$$

Differentiating with respecf to φ, we get the following equation for the perihelion motion

(2.10) $$\frac{d^2u}{d\varphi^2} + u\left(1 - \frac{6m^2}{h^2}\right) = \frac{m}{h^2}.$$

Solving (2.10), we get

(2.11) $$u = \frac{m}{h^2}\left[1 + e \cos\left(1 - \frac{3m^2}{h^2}\right)\varphi\right].$$

Using the relation

(2.12) $$h^2 = ma\,(1 - e^2)$$

in (2.11), we get the well known formula of INFELD & PLEBANSKI [2, 147]

(2.13) $$r = \frac{a\,(1 - e^2)}{1 + e \cos\left(1 - \frac{3m}{a\,(1 - e^2)}\right)\varphi'}$$

which represents the perihelion motion of a planet.

3. **Deflection of Light.** For the motion of light

(3.1) $$ds = 0.$$

Using (4.1) in (3.3), we have

(3.2) $$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2 = 1 + \frac{4m}{r},$$

where square and higher powers of the small quantity m/r are neglected.

Using (2.7) and (2.8) in (2.2), it reduces to

(3.3) $$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{1+4mu}{h^2}.$$

Neglecting the small relativistic expression $4mu$, from (3.3), we have

(3.4) $$u = \frac{\cos\varphi}{h}.$$

(3.4) gives that the path of light ray is a straight line. Let the distance of the light ray from the sun be R; from (3.4), we have

(3.5) $$h = R.$$

Differentiating (3.3) with respect to φ and using (3.5), we have

(3.6) $$\frac{d^2u}{d\varphi^2} + u = \frac{2m}{R^2}.$$

Solving (3.6), we get

(3.7) $$u = \frac{2m}{R^2}\left(1 + \frac{R}{2m}\cos\varphi\right).$$

In rectangular coordinates $x = r\cos\varphi = (\cos\varphi)/u$ and $y = r\sin\varphi = (\sin\varphi)/u$, (3.7) takes the form

(3.8) $$x = R - \frac{2m}{R}\sqrt{x^2+y^2}.$$

The second term measures the very slight deviation from the straight line $x = R$. The asymptotes are found by taking y very large compared with x. (3.8) then becomes

(3.9) $$x = R - \frac{2m}{R}(\pm y).$$

Therefore, the small angle between the asymptotes is $4m/R$, which is a well known result due to EDDINGTON [1, 97].

REFERENCES

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ÖZET

Bu yazıda, Genel Relativite Teorisinden, perihelion hareketi ve ışığın defleksiyonu için bilinen ifadeler, yaklaşım metodu ile elde edilmektedir.