ON THE DERIVATIVES OF THE MEAN VALUES OF INTEGRAL FUNCTIONS REPRESENTED BY DIRICHLET SERIES

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The definition of the mean value of an integral function represented by a DIRICHLET series is extended to the derivatives of the given function. The paper then studies some properties of the mean functions thus obtained as well as those of their derivatives.

1. Consider the DIRICHLET series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda}$$

where

$$\lambda_{n+1} > \lambda_n, \ \lambda_n, \ \lambda_1 \geq 0, \ \lim_{n\to\infty} \ \lambda_n = \infty, \ s = \sigma + it$$

and

(1.1) $$\lim_{n\to\infty} \sup \frac{\log n}{\log \lambda_n} = E < \infty.$$

Let σ_c and σ_a be the abscissa of convergence and abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c = \infty$ then σ_a will also be infinite, since according to a known result [1, 4] a DIRICHLET series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence and so $f(s)$ is an integral function.

The Mean Value of $f(s)$ is

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(1.2) $$I_2(\sigma) = I_2(\sigma, f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f(\sigma + it) |^2 \, dt.$$

Extending this definition to $f^{(p)}(s)$, the p-derivative of $f(s)$, we set

(1.3) $$I_2(\sigma, f^{(p)}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} | f^{(p)}(\sigma + it) |^2 \, dt.$$

Let

(1.4) $$\begin{cases} \mu(\sigma) = \max\limits_{n\geq 1} . \{ | a_n | e^{\sigma\lambda_n} \} \\ M(\sigma) = \text{l. u. b.}_{-\infty<t<\infty} | f(\sigma + it) | \end{cases},$$

be respectively the maximum term and maximum modulus of an integral function $f(s)$. Let $\lambda_{\nu(\sigma)}$ be the λ_n corresponding to the maximum term of the series of $f(s)$ for $Re(s) = \sigma$. Then evidently $\lambda_{\nu(\sigma)}$ is a non-decreasing function and $\nu(\sigma)$ is called the rank of the maximum term $\mu(\sigma)$. It is known [3, 67] that

(1.5) $$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)} \, dt.$$

It is also known* [2, 523] that for functions of finite non-zero linear order ϱ and lower order λ,

(1.6) $$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log \log I_2(\sigma)}{\sigma} = \frac{\varrho}{\lambda}.$$

Throughout this paper we shall assume that the function $f(s)$ is of finite non-zero linear order and satisfies (1.1).

The object of this paper is to study some properties of $I_2(\sigma)$ and its derivative and also of $I_2(\sigma, f^{(m)})$ and its derivative.

2. Theorem 1. *If*

(2.1) $$\lim_{\sigma\to\infty} {\sup \atop \inf} \frac{\log I_2(\sigma)}{e^{\varrho\sigma}} = \frac{T}{t}.$$

and

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* Result (1.6) has been proved under the condition

$$\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D = 0,$$

though the result also holds for $E < \infty$.

(2.2) $$\frac{I_2'(\sigma)}{I_2(\sigma)} \sim \alpha\, e^{\varrho\sigma}$$

for large values of σ, *where* α *is a positive constant* $o < f < T < \infty$, *then*

(*i*) $f(s)$ *is of linear regular growth,*

(2.3) (*ii*) $\varrho T = \varrho t = \alpha$;

and

(2.4) (*iii*) $$\lim_{\sigma\to\infty} \frac{I_2'(\sigma)}{I_2(\sigma)\log I_2(\sigma)} = \varrho.$$

Proof : (*i*) From (2.2), we have

$$\log \left\{ \frac{I_2'(\sigma)}{I_2(\sigma)} \right\} \sim \varrho\sigma + \log \alpha.$$

Hence

$$\lim_{\sigma\to\infty} \left[\frac{\log \frac{I_2'(\sigma)}{I_2(\sigma)}}{\sigma} \right] = \varrho,$$

and so $f(s)$ is of linearly regular growth.

(*ii*) We know [4]

(2.5) $$\log I_2(\sigma) \equiv \log I_2(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{I_2'(x)}{I_2(x)}\, dx.$$

From (2.5) and (2.2), we have

$$\log \left\{ \frac{I_2(\sigma)}{I_2(\sigma_0)} \right\} \sim \alpha \int_{\sigma_0}^{\sigma} e^{\varrho x}\, dx$$

$$\sim \frac{\alpha}{\varrho}\left(e^{\varrho\sigma} - e^{\varrho\sigma_0}\right).$$

Hence

(2.6) $$\lim_{\sigma\to\infty} \left\{ \frac{\log I_2(\sigma)}{e^{\varrho\sigma}} \right\} = \frac{\alpha}{\varrho}\ .$$

Then from (2.1) and (2.6), we get (2.3).

(*iii*) From (2.2), given any $\varepsilon > 0$, we can find σ_0 such that for $x \geq \sigma_0$,

$$(\alpha - e)\, e^{\varrho x} < \frac{I_2'(x)}{I_2(x)} < (\alpha + \varepsilon)\, e^{\varrho a}.$$

Hence

$$(\alpha-\varepsilon)\int_{\sigma_0}^{\sigma} e^{\varrho x}\,dx < \int_{\sigma_0}^{\sigma} \frac{I_2'(x)}{I_2(x)}\,dx < (\alpha+\varepsilon)\int_{\sigma_0}^{\sigma} e^{\varrho x}\,dx.$$

Since the integrands are positive increasing functions, we have

$$\frac{\alpha-\varepsilon}{\varrho}\;\frac{e^{\varrho\sigma}-e^{\varrho\sigma_0}}{\left\{\dfrac{I_2'(\sigma)}{I_2(\sigma)}\right\}} < \frac{\log I_2(\sigma)}{\left\{\dfrac{I_2'(\sigma)}{I_2(\sigma)}\right\}} - \frac{\log I_2(\sigma_0)}{\left\{\dfrac{I_2'(\sigma)}{I_2(\sigma)}\right\}} < \frac{\alpha+\varepsilon}{\varrho}\;\frac{e^{\varrho\sigma}-e^{\varrho\sigma_0}}{\left\{\dfrac{I_2'(\sigma)}{I_2(\sigma)}\right\}}.$$

Hence, on taking limits and using (2.2) we get (2.4).

3. **Theorem 2.** *Let $f(s)$ be an integral function of linear order δ and lower order λ. If $I_2'(\sigma, f^{(m)})$ is the derivative of $I_2(\sigma, f^{(m)})$ and $\lambda \geq \delta > 0$, then*

(3.1) $$I_2(\sigma, f) \leqq I_2'(\sigma, f) \leqq 4\, I_2'(\sigma, f^{(1)}) \leqq 4^2\, I_2'(\sigma, f^{(2)}) \leqq \cdots \leqq 4^m\, I_2'(\sigma, f^{(m)}).$$

almost everywhere for $\sigma > o_0 \geq 0$.

In order to prove this theorem we need the following lemma:

Lemma: *Let $f(s)$ be an integral function of linear order ϱ and lower order λ. If $\lambda \geq \delta > 0$, then*

(3.2) $$I_2(\sigma, f) \leqq I_2'(\sigma, f).$$

Proof of the lemma: We know [4] that

(3.3) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left[\frac{\log\left\{\dfrac{I_2'(\sigma)}{I_2(\sigma)}\right\}}{o}\right] = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Hence

(3.4) $$I_2(\sigma)\, e^{o(\lambda-\delta)} < I_2'(\sigma) < I_2(\sigma)\, e^{\sigma(\varrho+\varepsilon)}$$

almost everywhere for $o > o_i \geq 0$ and ε being any positive number. For $\lambda \geq \delta > 0$, from (3.4) we get (3.2).

Proof of the theorem: We have

(3.5) $$I_2(o, f^{(1)}) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f^{(1)}(\sigma+tt)|^2\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\lim_{\varepsilon\to 0} \frac{f(\sigma+it) - f(\sigma(1-\varepsilon)+it)}{\varepsilon\sigma}\right|^2 di$$

$$\geq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \lim_{\varepsilon\to 0} \left\{\frac{|f(\sigma+it)|^2 - 2|f(\sigma+it)|\,|f(\sigma(1-\varepsilon)+it)| + |f(\sigma(1-\varepsilon)+it)|^2}{\varepsilon^2\sigma^2}\right\} dt$$

From SCHWARZ's inequality, we have

$$(3.6) \qquad \frac{1}{2T}\int_{-T}^{T} 2\,|\,f(\sigma+it)\,|\;|\,f(\sigma(1-\varepsilon)+it)\,|\;dt \leqq$$

$$\frac{1}{2T}\,2\left\{\int_{-T}^{T}|\,f(\sigma+it)\,|^2\,dt\int_{-T}^{T}|\,f(\sigma(1-\varepsilon)+it)\,|^2\,dt\right\}^{1/2}.$$

Applying (3.6) in (3.5) and taking the limit outside the integral, which is justified since the integrals are uniformly convergent, we get

$$I_2(\sigma, f^{(1)}) \geq \lim_{\varepsilon\to 0}\left[\frac{\{I_2(\sigma)\}^{1/2}-\{I_2(\sigma(1-\varepsilon))\}^{1/2}}{\varepsilon\,\sigma}\right]^2$$

$$=\left[\frac{d}{d\sigma}\{I_2(\sigma)\}^{1/2}\right]^2$$

$$=\frac{1}{4}\;\frac{\{I_2'(\sigma)\}^2}{I_2(\sigma)}.$$

Using the above lemma, we get

$$I_2'(\sigma) \leqq 4\,I_2(\sigma, f^{(1)})$$

almost everywhere for $\sigma > \sigma_1 \geq 0$.

Similar results can be deduced for the higher derivatives and hence the theorem follows for almost all values of $\sigma > \sigma_0 \geq 0$, where

$$\sigma_0 = \max.\ (\sigma_l\,,\ \sigma_1\,,\ldots,\ \sigma_m).$$

Theorem 3. *For almost all values of* $\sigma > \sigma_0$,

$$I_2(\sigma)\;e^{\sigma(\lambda-\varepsilon)} < I_2'(\sigma) < I_2(\sigma)\;e^{\sigma(\varrho+\varepsilon)}.$$

Further, if

$$(3.7) \qquad \lim_{\sigma\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{I_2'(\sigma)\,/\,I_2(\sigma)}{e^{\varrho\sigma}}\right\} = \begin{matrix}\alpha\\ \beta\end{matrix}\quad,\ \textit{then}$$

$$I_2(\sigma)\,\{(\beta-\varepsilon)\;e^{\varrho\sigma}\} < I_2'(\sigma) < I_2(\sigma)\,\{(\alpha+\varepsilon)\;e^{\varrho\sigma}\}$$

where ε *is an arbitrary small positive number.*

Proof. From (3.3), for any $\varepsilon > 0$ and $\sigma > \sigma_0$

$$e^{\sigma(\lambda-\varepsilon)} < \frac{I_2'(\sigma)}{I_2(\sigma)} < e^{\sigma(\varrho+\varepsilon)},$$

therefore,

$$I_2(\sigma)\ e^{\sigma(\lambda-\varepsilon)} < I_2'(o) < I_2(\sigma)\ e^{\sigma(\varrho+\varepsilon)}\ .$$

Again, if (3.7) holds, then for any $\varepsilon > 0$ and $\sigma > \sigma_0$

$$(\beta-\varepsilon)\ e^{\varrho\sigma}\ I_2(\sigma) < I_2'(\sigma) < (\alpha+\varepsilon)\ e^{\varrho\sigma}\ I_2(\sigma).$$

We now prove the following lemmas:

4. **Lemma 1.** *Let $f(s)$ be an integral function of linear order ϱ and lower order λ. Then at the points of existence of $I_2'(\sigma)$ and for any $\varepsilon > 0$,*

(4.1) $$I_2'(\sigma) > \frac{I_2(\sigma)\ \log I_2(\sigma)}{(1-\lambda\,/\,(\varrho+\varepsilon))\ \sigma}\ ,$$

and

(4.2) $$I_2'(\sigma) > \frac{I_2(\sigma)\ \log I_2(\sigma)}{(1/\lambda-1/(\varrho+\varepsilon))\ \log \lambda_{\nu(\sigma)}}$$

for almost all values of $\sigma > \sigma_0$.

Proof. We know [4]

(4.3) $$\lim_{\sigma\to\infty} \sup \left\{\frac{\log I_2(\sigma)}{\sigma\,\lambda_{\nu(\sigma)}}\right\} \leq 2\,(1-\lambda/\varrho)$$

and

(4.4) $$\lim_{\sigma\to\infty} \sup \left\{\frac{\log I_2(\sigma)}{\lambda_{\nu(\sigma)}\ \log \lambda_{\nu(\sigma)}}\right\} \leq 2\,(1/\lambda-1/\varrho).$$

For any $\varepsilon < 0$ and for sufficiently large σ, the inequality (4.3) gives

(4.5) $$\log I_2(\sigma) < 2\,(1-\lambda/(\varrho+\varepsilon))\ \sigma\ \lambda_{\nu(\sigma)} = 2\,(1-\lambda/(\varrho+\varepsilon))\ \sigma\ \frac{\mu'(\sigma)}{\mu(\sigma)}\ ,$$

since

$$\mu(\sigma) = |\,a_{\nu(\sigma)}\,|\ e^{\sigma\,\lambda\nu(\sigma)}\ ,$$

therefore,

$$\frac{\mu'(\sigma)}{\mu(\sigma)} = \lambda_{\nu(\sigma)}\ ,$$

for almost all $\sigma \geq o_0$.

Now

$$I_2(\sigma) = \sum_{n=1}^{\infty} |\,a_n\,|^2\ e^{2\sigma\lambda n} \geq \{\,\mu(\sigma)\,\}^2\ .$$

Hence

$$\frac{\mu'(\sigma)}{\mu(\sigma)} \leq \frac{1}{2} \frac{I_2'(\sigma)}{I_2(\sigma)} . \tag{4.6}$$

Using (4.6) in (4.5), we have

$$I_2'(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{(1-\lambda/(\varrho+\varepsilon))\,\sigma} .$$

Similarly using (4.4), we can obtain (4.2).

Lemma 2. *If $f(s)$ is of linear regular growth, then*

$$I_2'(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{\varepsilon\,\sigma} .$$

and

$$I_2'(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{\varepsilon\,\lambda_{\nu(\sigma)}} .$$

for almost all $\sigma > \sigma_0$ and ε being any arbitrary small positive number.

These follow immediately from (4.1) and (4.2) on putting $\varrho = \lambda$.

Lemma 3. *Let $f(s)$ be an integral function of linear order ϱ and lower order λ, then*

$$I_2'(\sigma) > I_2(\sigma) \left\{ \frac{\log I_2(\sigma)}{(1/\lambda - 1/(\varrho+\varepsilon))(\varrho+\varepsilon)\sigma} \right\} . \tag{4.7}$$

for almost all $\sigma > \sigma_0$, $\varepsilon > 0$.

Proof. We know that

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log \lambda_{\nu(\sigma)}}{\sigma} \right\} = \begin{matrix}\varrho\\ \lambda\end{matrix} , \tag{4.8}$$

provided that (1.1) holds.

Using (4.8) in (4.2), we get (4.7).

Theorem 4. *Let $f(s)$ be an integral function of linear order ϱ and lower order λ. If $\lambda \geq \delta > 0$ then for $s > 0$ and $m = 1, 2, \ldots$*

$$I_2'(\sigma, f^{(m)}) > I_2'(\sigma) \left\{ \frac{\log I_2'(\sigma)}{(1+\varepsilon)\,\sigma} \right\}^m , \tag{4.9}$$

$$I_2'(\sigma, f^{(m)}) > I_2'(\sigma) \left\{ \frac{\log I_2'(\sigma)}{(1-\lambda/(\varrho+\varepsilon))\,\sigma} \right\}^m , \text{ and} \tag{4.10}$$

$$I_2'(\sigma, f^{(m)}) > I_2'(\sigma) \left\{ \frac{\log I_2'(\sigma)}{(1/\lambda - 1/(\varrho+\varepsilon))\,\sigma \log \lambda_{\nu(\sigma, f^{(m)})}} \right\}^m \tag{4.11}$$

for almost all values of $\sigma > \sigma_0 \geq 0$.

Proof. We know [4] that

$$I_2'(\sigma) > \frac{I_2(\sigma)\log I_2(\sigma)}{(1+\varepsilon)\sigma}. \tag{4.12}$$

for $\sigma > \sigma_0'$ and $\varepsilon > 0$.

Writing (4.12) for $f^{(1)}(s)$, we have for $\sigma > \sigma_1'$,

$$I_2'(\sigma, f^{(1)}) > \frac{I_2(\sigma, f^{(1)})\log I_2(\sigma, f^{(1)})}{(1+\varepsilon)\sigma} \geq \frac{I_2'(\sigma)\log I_2'(\sigma)}{4(1+\varepsilon)\sigma}, \tag{4.13}$$

since from § 3 (Lemma) $I_2(\sigma, f^{(1)}) \geq \frac{1}{4} I_2'(\sigma)$.

Now writing (4.13) for $f^{(p)}(s)$, we have

$$\frac{I_2'(\sigma, f^{(p)})}{I_2(\sigma, f^{(p-1)})} > \frac{\log I_2'(\sigma, f^{(p-1)})}{4(1+\varepsilon)\sigma}.$$

for $\sigma > \sigma_1^p$.

Taking $p = 1, 2, \dots, m$ and multiplying together, we get

$$I_2'(\sigma, f^{(m)}) > \frac{I_2'(\sigma)\prod_{p=1}^{m} \log I_2'(\sigma, f^{(p-1)})}{\{4(1+\varepsilon)\sigma\}^m}.$$

almost everywhere for $\sigma > \sigma_0 \geq 0$, where

$$\sigma_0 = \max.(\sigma_1, \sigma_2, \dots, \sigma_{m-1}, \sigma_1', \sigma_2', \dots, \sigma'_m).$$

But if $\lambda \geq \delta > 0$, we have from the Lemma of Theorem 2

$$I_2'(\sigma, f) \leq 4 I_2'(\sigma, f^{(1)}) \leq \dots \leq 4^m I_2'(\sigma, f^{(m)}).$$

Hence

$$I_2'(\sigma, f^{(m)}) > I_2'(\sigma)\left\{\frac{\log I_2'(\sigma)}{(1+\varepsilon)\sigma}\right\}^m.$$

To prove (4.10), write (4.1) for $f^{(1)}(s)$, we have

$$I_2(\sigma, f^{(1)}) > \frac{I_2(\sigma, f^{(1)})\log I_2(\sigma, f^{(1)})}{(1-\lambda/(\varrho+\varepsilon))\sigma} \geq \frac{I_2'(\sigma)\log I_2'(\sigma)}{4(1-\lambda/(\varrho+\varepsilon))\sigma}.$$

So for the p^{th}-derivative,

$$\frac{I_2'(\sigma, f^{(p)})}{I_2'(\sigma, f^{(p-1)})} > \frac{\log I_2'(\sigma, f^{(p-1)})}{4(1+\varepsilon)\sigma}.$$

for almost all $\sigma > \sigma_1^p$.

Taking $p = 1, \dots, m$ and multiplying the m inequalities thus obtained and proceeding as in (4.9), leads to (4.10).

To prove (4.11), start with (4.2) and proceed as above.

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ÖZET

Bir DIRICHLET serisi ile gösterilen bir fonksiyonun ortalama değeri, tanımı söz konusu fonksiyonun türevlerine de uygulandıktan sonra elde edilen bu ortalama değer fonksiyonları ile türevleri hakkında bazı sonuçlar ispat edilmektedir.