AN AXIOMATIC CHARACTERIZATION OF THE REDUCED HOMOLOGY THEORY

PATRICK SHANAHAN and J. W. T. YOUNGS (*)

It is the purpose of this paper to provide an axiomatic approach to reduced homology (and cohomology) theory. A *reduced homology theory* is *defined* to be a triple $(\tilde{H}, \sim, \tilde{\partial})$ satisfying seven axioms analogous to the well known axioms of EILENBERG and STEENROD for *homology*. Given a homology theory H, EILENBERG and STEENROD have a construction R which provides $R(H)$, a reduced homology theory in their sense. It is shown that $R(H)$ satisfies the axioms for a reduced homology theory given in this paper. Moreover, a second construction S is developed, which may be applied to any reduced homology theory to yield a homology theory. The characterization of the reduced theory is completed by showing that the constructions S and R are inverse to each other, in an appropriate sense. In fact, these constructions provide a one-one correspondence between classes of isomorphic homology theories and classes of isomorphic reduced homology theories, and thus either type of system can be taken as a starting point in the algebraic study of topological spaces. As an application, it can be shown that if a homology theory satisfies the continuity axiom, then so does the corresponding reduced theory, and conversely. Thus the continuity axiom serves to characterize the reduced CECH homology theory. Remarks similar to those above hold for cohomology theory.

Introduction

1. The axiomatic approach to homology theory developed by EILENBERG and STEENROD (see [1] and [2]) has stimulated similar efforts in other areas of algebraic topology; for example in homotopy theory [see [3]). The object of this paper is to present axioms for reduced homology theory. As a consequence of the investigation one obtains further insight on the connection between reduced and ordinary theories. For example, there is a natural one-to-one correspondence between them formalized in 4.1.

Without attempting to labor the point herein, it appears fair to state that, from the point of view of computations, the reduced theory offers several advantages which make it all the more desirable to exhibit an axiomatic approach. For example, such a development makes unnecessary the extraneous introduction of special groups in the computations of the homology groups of spheres by induc-

(*) The work towards this paper was supported in part by *NSF* grant G-2256.

tion (see [1], p. 46). Moreover, it will be shown elsewhere that whereas it is a difficult task to obtain non-trivial examples of an ordinary theory this is somewhat surprisingly not the case for a reduced theory.

2. For notational convenience the reader is referred to [1], pp. 3—14. Generally speaking, a familiarity with [1], Chapter 1, is assumed.

Definition 1. *A reduced homology theory* $\underline{\underline{\tilde{H}}}$ is a triple $\{\tilde{H}, \sim, \tilde{\partial}\}$ of *three functions* which satisfies certain axioms given below.

The function $\tilde{H}$ is defined for each triple (q, X, A), where X and A are topological spaces (or compact spaces) with $X \supset A$ and q is an integer; its value is an abelian group $\tilde{H}_q(X, A)$.

Given a mapping $f:(X, A) \longrightarrow (Y, B)$ and an integer q, the function $\sim$ determines a homomorphism $\tilde{f}_q : \tilde{H}_q(X, A) \longrightarrow \tilde{H}_q(Y, B)$.

The function $\tilde{\partial}$ is defined for each triple (q, X, A) and its value is a homomorphism $\tilde{\partial}(q, X, A) : \tilde{H}_q(X, A) \longrightarrow \tilde{H}_{q-1}(A)$. (For convenience, $\tilde{H}_q(X, A)$ is written $\tilde{H}_q(X)$ if A is the empty set $\square$.)

The values of the last two functions are referred to as the *induced homomorphism* and the *boundary homomorphism* respectively, and as a rule will be denoted simply by $\tilde{f}$ and $\tilde{\partial}$.

To constitute a *reduced homology theory*, the functions must satisfy the following axioms:

$\tilde{A}_1$. If i is the identity map on a pair (X, A), then $\tilde{i}$ is the identity homomorphism on the group $\tilde{H}_q(X, A)$.

$\tilde{A}_2$. If $f:(X, A) \longrightarrow (Y, B)$, $g:(Y, B) \longrightarrow (Z, C)$, and $(gf):(X, A) \longrightarrow (Z, C)$ is the composition of f and g, then for each integer q the homomorphism $\widetilde{(gf)}$ induced by (gf) is the composition of the homomorphisms $\tilde{f}$ and $\tilde{g}$.

$\tilde{A}_3$. If $f:(X, A) \longrightarrow (Y, B)$ and $(f/A): A \longrightarrow B$ is the restriction of f to A, then the following diagram is *commutative* for each integer q, that is, $\tilde{\partial}\tilde{f} = \widetilde{(f/A)}\,\tilde{\partial}$.

$$\begin{array}{ccc}
\tilde{H}_q(X, A) & \xrightarrow{\ \tilde{f}\ } & \tilde{H}_q(Y, B) \\
\Big\downarrow \tilde{\partial} & & \Big\downarrow \tilde{\partial} \\
\tilde{H}_{q-1}(A) & \xrightarrow{\ \widetilde{(f/A)}\ } & \tilde{H}_{q-1}(B)
\end{array}$$

$\tilde{A}_4$. Given (X, A), if $i: A \longrightarrow X$, $j: X \longrightarrow (X, A)$ are inclusion maps, the *homology sequence*

$$\cdots \xrightarrow{\tilde{\partial}} \tilde{H}_q(A) \xrightarrow{\tilde{i}} \tilde{H}_q(X) \xrightarrow{\tilde{j}} \tilde{H}_q(X, A) \xrightarrow{\tilde{\partial}} \tilde{H}_{q-1}(A) \xrightarrow{\tilde{i}} \cdots$$

is *exact;* that is, the image of any group under a given homomorphism is the same as the kernel of the succeeding homomorphism.

$\tilde{A}_5$. If the maps $f_0: (X, A) \longrightarrow (Y, B)$ and $f_1: (X, A) \longrightarrow (Y, B)$ are homotopic, then $\tilde{f}_0 = \tilde{f}_1$ for each integer q.

$\tilde{A}_6$. If U is an open set such that $\bar{U} \subset A^0 \subset A \subset X$ [1], then the homomorphism $\tilde{e}$ induced by the inclusion map $e: (X - U, A - U) \longrightarrow (X, A)$ is an *isomorphism into* for each integer q; furthermore, if either $U \neq A$ or $q \neq 0$ then $\tilde{e}$ is *onto*.

$\tilde{A}_7$. If P is a point, $\tilde{H}_q(P) = 0$ for each integer q.

For easy reference the definition of a homology theory is also recorded. (See [1], pp. 10—12).

3. **Definition 2.** *A homology theory H* is a triple $\{H, *, \partial\}$ of three functions, precisely analagous to those of the previous definition, which satisfies the following seven axioms.

Axioms A_1 through A_5 are simply the first five axioms of Definition 1, with $\tilde{H}$, $\sim$, and $\tilde{\partial}$ replaced throughout by H, $*$, and ∂. (The $*$ is employed as a subscript.)

A_6. If U is an open set such that $\bar{U} \subset A_0 \subset A \subset X$ then the homomorphism e_* induced by the inclusion map $e: (X - U, A - U) \longrightarrow (X, A)$ is an isomorphism for each integer q.

A_7. If P is a point, then $H_q(P) = 0$ for each integer $q \neq 0$.

4. **Remark.** The role of the index q deserves some mention. Suppose k is an integer different from zero. Given a reduced homology theory $\underline{\underline{\tilde{H}}}$ consider a new triple of functions obtained from those given in $\underline{\underline{\tilde{H}}}$ by «shifting» the index an amount k; that is, associate with each integer q the functions given in $\underline{\underline{\tilde{H}}}$ for $(q + k)$. The new triple of functions satisfies all the axioms except $\tilde{A}_6$. Using the same device on a homology theory produces a triple of functions satisfying all the axioms except A_7. Hence it is not possible to obtain a spectrum of new theories

(1) The symbol $\bar{U}$ represents the closure of U, while A^0 is the interior of A.

from any given theory by this method; moreover, this suggests that the special role of A_7 in a homology theory is played by $\tilde{A}_6$ in a reduced homology theory.

5. Given a homology theory $\underline{\underline{H}}$, EILENBERG and STEENROD obtain a *constructed reduced theory* $R(\underline{\underline{H}})$ by means of a process R (see [1], 1.1 — 1.3) applied to $\underline{\underline{H}}$. There is no *a priori* guarantee that there is a connection between $R(\underline{\underline{H}})$ and a *reduced theory* in the sense of this paper; however, it is natural to conjecture that not only is $R(\underline{\underline{H}})$ a reduced theory, but that all reduced theories can be obtained in this manner.

Parallel to the device R of EILENBERG and STEENROD, a process S is developed in 2.1—2.5 which, given a reduced theory $\underline{\underline{\tilde{H}}}$, provides a *constructed homology theory* $S(\underline{\underline{\tilde{H}}})$. Again, there is no guarantee that there is a connection between $S(\underline{\underline{\tilde{H}}})$ and a *homology theory* in the sense of EILENBERG and STEENROD, much less that all homology theories can be obtained in this manner. Once more, however, it is natural to conjecture that this is in fact the case.

One of the objects of this paper is to show that these conjectures are true; specifically:

I. If $\underline{\underline{H}}$ is a *homology theory* then the constructed theory $R(\underline{\underline{H}})$ of EILENBERG and STEENROD is a reduced homology theory; that is, satisfies $\tilde{A}_1$ through $\tilde{A}_7$.

II. If $\underline{\underline{\tilde{H}}}$ is a reduced homology theory then the constructed theory $S(\underline{\underline{\tilde{H}}})$ is a homology theory; that is, satisfies A_1 through A_7.

III. The constructions of I and II are inverse to each other; that is,

(*a*) $S\{R(\underline{\underline{H}})\}$ is isomorphic to $\underline{\underline{H}}$,

(*b*) $R\{S(\underline{\underline{\tilde{H}}})\}$ is isomorphic to $\underline{\underline{\tilde{H}}}$.

It will follow from these results that any reduced homology theory $\underline{\underline{\tilde{H}}}$ is constructible from a homology theory $\underline{\underline{H}}$ which is *unique* up to isomorphism, and conversely, that any homology theory $\underline{\underline{H}}$ is constructible from a reduced homology theory $\underline{\underline{\tilde{H}}}$ which is *unique* in the same sense.

It is characteristic of the paper that the difficulty lies in *devising* the construction S and then *selecting* the isomorphisms which prove III (*a*) and III (*b*). The proofs, once these choices have been made, present no great obstacles.

1. The construction R

1.1 Given a homology theory $\underline{\underline{H}} = \{H, *, \partial\}$ the *constructed reduced theory* $R(H)$ is a triple $R(H) = \{\tilde{H}, \sim, \tilde{\partial}\}$ obtained from $\underline{\underline{H}}$ by considering a fixed point P_0 and the *collapsing map*

(1) $$\varphi : (X, A) \longrightarrow (P_0, \varphi(A))$$

defined by the functional relationship

$$\varphi(x) = P_0 \qquad x \varepsilon X.$$

(See [1], pp. 18-22).

Note that consideration of the set $\varphi(A)$ unifies the structure of the definition so that the cases $A = \square$ (hence $\varphi(A) = \square$) and $A \neq \square$ (whence $\varphi(A) = P_0$) need not be considered separately.

Consider

(2) $$\varphi_* : H_q(X, A) \longrightarrow H_q(P_0, \varphi(A))$$

and *define*

(3) $$\tilde{H}_q(X, A) = \operatorname{Ker} \varphi_*,$$

hence $\tilde{H}_q(X, A)$ is a subgroup of $H_q(X, A)$ and it is possible to consider the *inclusion homomorphism*

(4) $$K : \tilde{H}_q(X, A) \longrightarrow H_q(X, A).$$

1.2 If X is any space, it is a consequence of axioms A_1 and A_4 (exactness) that $H_q(X, X)$ is zero for all q (see [1], Lemma 8.1, p. 20); in particular, $H_q(P_0, P_0) = 0$ for all integers q. Moreover, if $q \neq 0$ axiom A_7 states that $H_q(P_0) = 0$. *Thus if either $A \neq \square$ or $q \neq 0$ the homomorphism φ_* of (2) is trivial so that*

$$K : \tilde{H}_q(X, A) \longrightarrow H_q(X, A)$$

is the identity isomorphism.

1.3 If $f : (X, A) \longrightarrow (Y, B)$ it is an immediate consequence of axiom A_2 that the induced homomorphism $f_* : H_q(X, A) \longrightarrow H_q(Y, B)$ carries $\tilde{H}_q(X, A)$ into $\tilde{H}_q(Y, B)$ so that an induced homomorphism

(5) $$\tilde{f} : \tilde{H}_q(X, A) \longrightarrow \tilde{H}_q(Y, B)$$

may be defined by restricting f_* to the subgroup $\tilde{H}_q(X, A)$. The effect of this is to make the following diagram commutative.

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{f_*} & H_q(Y,B) \\ \uparrow{\scriptstyle K} & & \uparrow{\scriptstyle K'} \\ \widetilde{H}_q(X,A) & \xrightarrow{\widetilde{f}} & \widetilde{H}_q(Y,B) \end{array}$$

Similarly, it follows from axiom A_3 that the boundary homomorphism maps $\widetilde{H}_q(X, A)$ into $\widetilde{H}_{q-1}(A)$, so that it is meaningful to define the homomorphism

(6) $$\widetilde{\partial} : \widetilde{H}_q(X, A) \longrightarrow \widetilde{H}_{q-1}(A)$$

to be the restriction of ∂ to $\widetilde{H}_q(X, A)$. Hence the diagram

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{\partial} & H_{q-1}(A) \\ \uparrow{\scriptstyle K} & & \uparrow{\scriptstyle K'} \\ \widetilde{H}_q(X,A) & \xrightarrow{\widetilde{\partial}} & \widetilde{H}_{q-1}(A) \end{array}$$

is commutative.

In this fashion, to each homology theory $\underline{H} = \{H, *, \partial\}$ there corresponds a triple $\underline{\widetilde{H}} = \{\widetilde{H}, \sim, \widetilde{\partial}\}$. As has already been indicated this *correspondence* is denoted by the letter R, so that $\underline{\widetilde{H}} = R(\underline{H})$.

1.4 Teorem 1. *The constructed theory $R(\underline{H})$ is a reduced homology theory.*

Proof. The verification of axioms $\widetilde{A}_1$ through $\widetilde{A}_5$ offers no difficulty and is left to the reader. The proofs that the remaining two axioms are satisfied are exhibited here by way of example.

Proof of $\widetilde{A}_6$. Let (X, A) be a pair and U be an open set such that $\overline{U} \subset A_0$. Consider the diagram

$$\begin{array}{ccc} H_q(X-U, A-U) & \xrightarrow{e_*} & H_q(X,A) \\ \uparrow{\scriptstyle K} & & \uparrow{\scriptstyle K'} \\ \widetilde{H}_q(X-U, A-U) & \xrightarrow{\widetilde{e}} & \widetilde{H}_q(X,A) \end{array}$$

where K and K' are the homomorphism of 1.1, and the homomorphism e_* is induced by an inclusion map. The definition of $\tilde{e}$ (see 1.3) makes the diagram commutative. Note that e_* is an isomorphism by A_6, whereas K is an inclusion. Hence $\tilde{e}$ is an isomorphism into. *If either* $U \neq A$ *or* $q = 0$, then by 1.2 the inclusion K is *onto*. Hence e is onto since K' is an inclusion.

Proof of $\tilde{A}_7$. Let P by any point and consider the sequence

$$P \xrightarrow{\varphi} P_0 \xrightarrow{\psi} P.$$

Since the composition $\psi\varphi$ is the identity map of P, it follows from axioms A_1 and A_2 that $\psi_*\varphi_*$ is the identity isomorphism on $H_q(P)$ for all integers q. Hence $\operatorname{Ker}\varphi_* = 0$, and by (3)

$$\tilde{H}_q(P) = 0 \qquad \text{for all integers } q.$$

2. The construction S

2.1 Given a reduced homology theory $\underline{\underline{\tilde{H}}}$, the object of this section is to develop the constructed homology theory $S(\underline{\underline{\tilde{H}}})$ mentioned in item II of the introduction. The basic idea possesses such fundamental simplicity that it will be revealed now before there is danger of it being smothered in technical details.

If a space has precisely two points P and Q, then the inclusion map $e : (P, \square) \longrightarrow (P \cup Q, Q)$ induces homomorphisms which by $\tilde{A}_6$ are isomorphisms if $q \neq 0$. But by $\tilde{A}_7$ the groups $\tilde{H}_q(P)$ are trivial; *hence* $\tilde{H}_q(P \cup Q, Q) = 0$ *for* $q \neq 0$. Note that this has tbe flavor of axiom A_7 for homology theory, if one regards $\tilde{H}_q(P \cup Q, Q)$ as the homology group of P. Iu short, defining $H_q(P)$ to be $\tilde{H}_q(P \cup Q, Q)$ provides the seed for a theory which satisfies A_7 (but *not* necessarily $\tilde{A}_7$). The interesting fact is that the device of adding an *isolated* point Q to P works with any space. Unfortunately, an identification procedure of some sort is necessary since there does not exist a fixed point disjoint from all spaces — the technique of direct limits, though it may appear cumbersome, actually eliminates troublesome conceptual complications.

2.2 Given (X, A), let $\mathfrak{F}(X, A)$ denote the family consisting of all pairs $(\underline{X}, \underline{A})$ such that

$$\text{(7)} \qquad \underline{X} - X = \underline{A} - A$$

is an isolated point of $\underline{X}$.

If $f : (X, A) \longrightarrow (Y, B)$, consider any $(\underline{X}, \underline{A}) \in \mathfrak{F}(X, A)$ and $(\underline{Y}, \underline{B}) \in \mathfrak{F}(Y, B)$. Let $\underline{f} : (\underline{X}, \underline{A}) \longrightarrow (\underline{Y}, \underline{B})$ denote the map defined by

(8) $$\underline{f}(x) = \begin{cases} f(x) & \text{if} \quad x \varepsilon X, \\ \underline{Y} - Y & \text{if} \quad x = \underline{X} - X. \end{cases}$$

If (X_1, A_1) and (X_2, A_2) are in $\mathfrak{F}(X, A)$ then the identity map

$$i : (X, A) \longrightarrow (X, A)$$

is uniquely extendable to a map from (X_1, A_1) to (X_2, A_2). In fact, the extension of i, designated by

$$i_1^2 : (X_1, A_1) \longrightarrow (X_2, A_2)$$

is obtained by defining the image of $X_1 - X$ to be $X_2 - X$. Though i_1^2 will not be the identity unless $X_2 - X = X_1 - X$, it is a homeomorphism, and as a direct consequence of $\widetilde{A}_1$ and $\widetilde{A}_2$ it induces *isomorphisms*

(9) $$\widetilde{i_1^2} : \widetilde{H}_q(X_1, A_1) \longrightarrow \widetilde{H}_q(X_2, A_2)$$

for each integer q. Moreover, by the same axioms

(10) $\widetilde{i_1^1}$ is the identity isomorphism

and

(11) $$\widetilde{i_1^3} = \widetilde{i_2^3}\, \widetilde{i_1^2}.$$

For each integer q consider the set $\underline{\underline{G}}_q$ of groups $\widetilde{H}_q(\underline{X}, \underline{A})$ for $(\underline{X}, \underline{A}) \varepsilon \mathfrak{F}(X, A)$. Direct $\underline{\underline{G}}_q$ trivially by defining

$$\widetilde{H}_q(X_1, A_1) < \widetilde{H}_q(X_2, A_2)$$

if $(X_1, A_1))$ and (X_2, A_2) are in $\mathfrak{F}(X, A)$. In view of (9), (10), and (11) one has a direct system of groups and homomorphisms. (See [1], Chapter VIII).

The *homology groups*

(12) $$\underline{H}_q(X, A)$$

is defined to be the *direct limit* of the system.

In view of (9) it is clear that the *projection*

(13) $$\eta_1 : \widetilde{H}_q(X_1, A_1) \longrightarrow \underline{H}_q(X, A)$$

is an *isomorphism*, and as usual

(14) $$\eta_1 = \eta_2\, \widetilde{i_1^2}.$$

2.3 Suppose that $(\underline{X}, \underline{A}) \varepsilon \mathfrak{F}(X, A)$ and $U = \underline{A} - A = \underline{X} - X$, an isolated point of $\underline{X}$ (see (7)). Clearly U is an open set and $\overline{U} \subset \underline{A}_0 \subset \underline{A} \subset \underline{X}$. Note that $(\underline{X} - U, \underline{A} - U) = (X, A)$ and hence the inclusion map $e: (X, A) \longrightarrow (\underline{X}, \underline{A})$ satisfies the conditions of $\widetilde{A}_6$. Therefore, if η is the projection isomorphism of (13), then the composition

$$\eta \tilde{e} : \widetilde{H}_q(X, A) \longrightarrow \underline{H}_q(X, A)$$

is an *isomorphism into*.

It is an easy consequence of (14) and $\widetilde{A}_2$ that $\tilde{e}$ is independent of the choice of $(\underline{X}, \underline{A})$ in $\mathfrak{F}(X, A)$. *If either* $A \neq \square$ *or* $q \neq 0$ *then* e *is an isomorphism so that in this case* $\eta \tilde{e}$ *maps* $\widetilde{H}_q(X, A)$ *isomorphically onto* $\underline{H}_q(X, A)$. (The reader will notice the similarity between 1.2 and 2.3).

2.4 Proceeding to the construction of *induced homomorphisms*, let

$$f: (X, A) \longrightarrow (Y, B)$$

be any map and consider any $(\underline{X}, \underline{A}) \varepsilon \mathfrak{F}(X, A)$ and $(\underline{Y}, \underline{B}) \varepsilon \mathfrak{F}(Y, B)$. The extension

$$\underline{f}: (\underline{X}, \underline{A}) \longrightarrow (\underline{Y}, \underline{B})$$

of f has been defined in (8). At the group level one has the following diagram where η and η' are the projection isomorphisms.

$$\begin{array}{ccc} \widetilde{H}_q(\underline{X}, \underline{A}) & \xrightarrow{\underline{f}} & \widetilde{H}_q(\underline{Y}, \underline{B}) \\ \downarrow \eta & & \downarrow \eta' \\ \underline{H}_q(X, A) & & \underline{H}_q(Y, B) \end{array}$$

Define the induced homomorphism $f_{*}: \underline{H}_q(X, A) \longrightarrow \underline{H}_q(Y, B)$ so as to achieve commutativity; that is,

(15) $$f_{*} = \eta' \underline{\tilde{f}} \eta^{-1}.$$

The fact that f_{*} is independent of the selection of $(\underline{X}, \underline{A})$ and $(\underline{Y}, \underline{B})$ is an easy consequence of (14) and axiom $\widetilde{A}_2$.

2.5 To define the *boundary homomorphism*, let $(\underline{X}, \underline{A}) \varepsilon \mathfrak{F}(\underline{X}, \underline{A})$ and let $i: \underline{A} \longrightarrow (\underline{A}, \square) = (\underline{A}, \underline{A} - A)$ be the inclusion map. At the group level one has the following diagram where η and ε are the projection isomorphisms.

$$\begin{array}{ccc} \tilde{H}_q(\underline{X},\underline{A}) & \xrightarrow{\quad\tilde{\partial}\quad} & \tilde{H}_{q-1}(\underline{A}) \\ & & \downarrow \tilde{i} \\ \downarrow \eta & & H_{q-1}(\underline{A},\square) \\ & & \downarrow \varepsilon \\ \underline{H}_q(X,A) & & \underline{H}_{q-1}(A) \end{array}$$

Define the boundary homomorphism $\underline{\partial}:\underline{H}_q(X,A)\longrightarrow \underline{H}_q(A)$ so as to achieve commutativity; that is

(16) $$\underline{\partial}=\varepsilon\,\tilde{i}\,\tilde{\partial}\,\eta^{-1}.$$

Remark. As will be seen by Lemma 1 of 3.3, the homomorphism $\tilde{i}$ is in fact an isomorphism; hence if we consider the composition $(\varepsilon\tilde{i})$ the diagram becomes a group theoretic copy of that in 2.4.

The task of showing that $\underline{\partial}$ is idependent of the selection of $(\underline{X},\underline{A})$ is an easy exercise in the use of $\tilde{A}_2$, $\tilde{A}_3$, and (14).

2.6 Theorem 2. *The constructed theory $\underline{\underline{S}}(\tilde{H})$ is a homology theory.*

Proof. In view of the fact that axioms A_1 through A_5 are identical with those for the reduced theory, and since the proof of axiom A_7 was essentially outlined in the introductory remarks to this section, it will suffice merely to record the proof of A_6; this will also serve to illuminate the construction described above.

Proof of A_6. Let (X,A) be a pair and let U be an open set such that $U\subset A^0$. If $(\underline{X},\underline{A})\,\varepsilon\,\mathfrak{F}(X,A)$, then clearly $(\underline{X}-U,\underline{A}-U)\,\varepsilon\,\mathfrak{F}(X-U,A-U)$ so that the diagram

$$\begin{array}{ccc} \tilde{H}_q(\underline{X}-U,\underline{A}-U) & \xrightarrow{\quad\tilde{e}\quad} & \tilde{H}_q(\underline{X},\underline{A}) \\ \downarrow \eta & & \downarrow \eta' \\ \underline{H}_q(X-U,A-U) & \xrightarrow{\quad e_{III}\quad} & \underline{H}_q(X,A) \end{array}$$

in which $\tilde{e}$ is induced by the inclusion map while η and η' are projection isomorphisms is meaningful. The diagram is commutative by the definition of e_{III}. (See 2.4.) Note that $\overline{U}\subset A^0\subset(\underline{A})^0$, and, moreover, $\underline{A}-U\supset\underline{A}-A\neq\square$. Thus the conditions of $\tilde{A}_6$ are satisfied and $\tilde{e}$ is an isomorphism for each integer q.

Since η and η' are isomorphisms it follows that the induced homomorphism e_* is an isomorphism for all q.

3. The characterization theorems

3.1 If $\underline{\underline{H}}$ a homology theory, then by Theorem 1 the constructed theory $R(\underline{\underline{H}})$ is a reduced homology theory. By Theorem 2 the entity $\underline{\underline{H}} = S\{R(\underline{\underline{H}})\}$ is a homology theory.

It will be shown that the «composition» construction SR reproduces the original theory; more precisely, that $S\{R(\underline{\underline{H}})\}$ is isomorphic to $\underline{\underline{H}}$. (See item III ($a$) of 0.5).

This means (see [1], pp. 118—119) that one must exhibit a family $\underline{\underline{h}}$ of isomorphisms $h(q, X, A): H_q(X, A) \longrightarrow \underline{H}_q(X, A)$ for each integer q and all pairs (X, A) such that the isomorphisms $h(q, X, A)$ commute with the induced homomorphisms and the boundary homomorphisms of $\underline{\underline{H}}$ and $\underline{\underline{\underline{H}}}$. For brevity, under these circumstances it is said that $\underline{\underline{h}}$ maps $\underline{\underline{H}}$ isomorphically onto $\underline{\underline{\underline{H}}}$.

As is usual in such situations, the real problem is to select the isomorphisms; the commutation properties follow by pedestrian arguments requiring endurance rather than ingenuity and are therefore omitted.

Let (X, A) be any pair and $(\underline{X}, \underline{A}) \varepsilon \mathfrak{F}(X, A)$. Consider the homomorphisms

$$e_* : H_q(X, A) \longrightarrow H_q(\underline{X}, \underline{A})$$

$$K : \tilde{H}_q(\underline{X}, \underline{A}) \longrightarrow H_q(\underline{X}, \underline{A})$$

$$\eta : \tilde{H}_q(\underline{X}, \underline{A}) \longrightarrow \underline{H}_q(X, A)$$

whare e_* is induced by the inclusion map, K is the inclusion homomorphism, and η is the projection of (13).

On writing $U = \underline{A} - A$, one observes that $\bar{U} = \underline{A} - A \subset A^0 \cup (\underline{A} - A) = (\underline{A})^0$. Hence by A_6

e_* is an isomorphism.

Moreover, $\underline{A}$ contains the point $\underline{A} - A$ and hence is not empty. Consequently, by 1.2

K is the identity homomorphism.

Finally, by (13) the projection

η is an isomorphism.

Define the family $\underline{\underline{h}}$ of isomorphisms $h(q, X, A)$ by

$$h(q, X, A) = \eta K^{-1} e_* . \tag{17}$$

Armed with this selection of isomorphisms, the commutativity properties mentioned above are readily demonstrable and one has

Theorem 3. *The family $\underline{\underline{h}}$ maps $\underline{\underline{H}}$ isomorphically onto $S\{R(\underline{\underline{H}})\}$.*

3.2. In this section let $\underline{\underline{\widetilde{H}}}$ be a reduced homology theory. Then $S(\underline{\underline{\widetilde{H}}})$ is a homology theory by theorem 2, and $\underline{\widetilde{H}} = R\{S(\underline{\underline{\widetilde{H}}})\}$ is a reduced homology theory by Theorem 1. It will be shown that the «composition» construction RS reproduces the original theory; more precisely, that $R\{S(\underline{\underline{\widetilde{H}}})\}$ is isomorphic to $\underline{\underline{\widetilde{H}}}$. As in 3.1 the problem is to select the family $\underline{\underline{k}}$ of isomorphisms

$$k(q, X, A) : H_q(X, A) \longrightarrow \widetilde{H}_q(X, A)$$

so that they commute with the induced homomorphisms and the boundary homomorphisms of $\underline{\underline{\widetilde{H}}}$ and $\underline{\underline{\underline{\widetilde{H}}}}$.

3.3. The selection of the family $\underline{\underline{k}}$ requires several lemmas, the first of which justifies the remark of 2.5.

Lemma 1. If $x \varepsilon X$, then $j : X \longrightarrow (X, x)$ induces isomorphisms

$$\widetilde{j} : \widetilde{H}_q(X) \approx \widetilde{H}_q(X, x)$$

for each integer q.

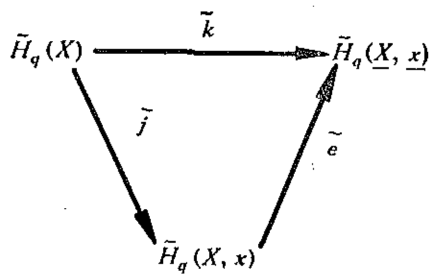
Proof. Consider the homology sequence of the pair (X, x), namely

$$\cdots \xrightarrow{\widetilde{\partial}} \widetilde{H}_q(x) \xrightarrow{\widetilde{i}} \widetilde{H}_q(X) \xrightarrow{\widetilde{j}} \widetilde{H}_q(X, x) \xrightarrow{\widetilde{\partial}} \widetilde{H}_{q-1}(x) \xrightarrow{\widetilde{i}} \cdots .$$

Since x is a point, $\widetilde{H}_q(x) = 0$ for all q by $\widetilde{A}_7$. Thus $\widetilde{i}$ and $\widetilde{\partial}$ are trivial, and the lemma follows as an easy consequence of $\widetilde{A}_4$.

Lemma 2. If $x \varepsilon X$ then the inclusion map $k : X \longrightarrow (\underline{X}, \underline{x})$ induces isomorphisms $\widetilde{k} : \widetilde{H}_q(X) \approx \widetilde{H}_q(\underline{X}, \underline{x})$ for each integer q.

Proof. Consider the following diagram



in which $\tilde{j}$ and $\tilde{e}$ are induced by inclusion maps. The diagram is commutative by $\tilde{A}_2$, and $\tilde{e}$ is an isomorphism by $\tilde{A}_6$ since in this application $A - \underline{x} \neq \underline{x} - x = U$. But $\tilde{j}$ is an isomorphism by Lemma 1, hence $\tilde{k}$ is an isomorphism.

(Only the case $q = 0$ of Lemma 2 is necessary in what follows.)

Remark. Given a pair (X, A) consider the collapsing map

$$\varphi : (X, A) \longrightarrow (P_0, \varphi(A))$$

of (1). If $(\underline{X}, \underline{A}) \in \mathfrak{F}(X, A)$ and $(P_0, \underline{\varphi(A)}) \in \mathfrak{F}(P_0, \varphi(A))$, then the extension

$$\underline{\varphi} : (\underline{X}, \underline{A}) \longrightarrow (\underline{P}_0, \underline{\varphi(A)})$$

of φ is defined in (8). Let

$$e : (X, A) \longrightarrow (\underline{X}, \underline{A})$$

be the inclusion map. It is important to eliminate a possible source of later confusion by observing that $(\underline{P}_0, \underline{\varphi(A)}) = (\underline{P}_0, \underline{\varphi}(\underline{A}))$. *These comments display information which is used in the remainder of this section.*

Lemmas 3 and 4 below concern themselves with a fundamental property of the homomorphisms induced by $\underline{\varphi}$ and e. The first concerns the special case in which $A = \square$ and hence $e : X \longrightarrow (\underline{X}, \underline{\square})$ and $\underline{\varphi} : (\underline{X}, \underline{\square}) \longrightarrow (\underline{P}_0, \underline{\square})$.

Lemma 3. *The sequence*

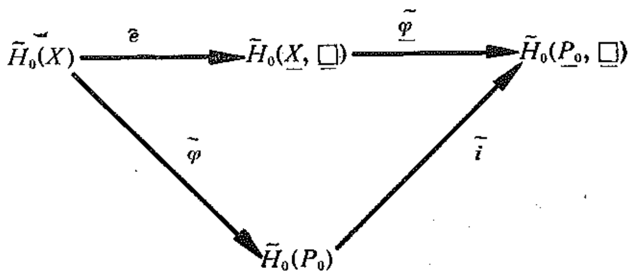
$$H_0(X) \xrightarrow{\ \tilde{e}\ } H_0(\underline{X}, \underline{\square}) \xrightarrow{\ \underline{\varphi}\ } H_0(\underline{P}_0, \underline{\square})$$

is exact. (See $\tilde{A}_4$.)

Remark. The lemma concerns itself with an aspect of the exceptional case of $\tilde{A}_6$; from that source one has $\operatorname{Ker} \tilde{e} = 0$. Lemma 3 provides the additional information that $\operatorname{Im} \tilde{e} = \operatorname{Ker} \underline{\varphi}$.

Proof. If X is empty, then $\tilde{H}_0(\underline{X}, \underline{\square}) = \tilde{H}_0(\underline{\square}, \underline{\square}) = 0$ (see 1.2), and the lemma is true. Assume $X \neq \square$.

Consider the diagram



where $\tilde{i}$ is induced by an inclusion map. Commutativity holds by $\tilde{A}_2$, and since $\tilde{H}_0(P_0) = 0$ by $\tilde{A}_7$, we have $\underline{\tilde{\varphi}}\,\tilde{e} = \tilde{i}\,\tilde{\varphi} = 0$, so that

$$\text{Im } \tilde{e} \subset \text{Ker } \underline{\tilde{\varphi}}. \tag{18}$$

It remains to show that Im $\tilde{e} \supset$ Ker $\underline{\tilde{\varphi}}$.

Let $x \in X$, and consider the triple (see [1], pp. 24—29)

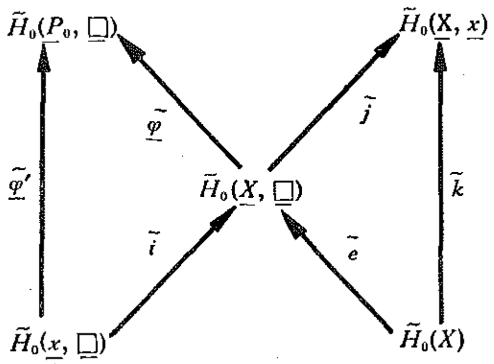
$$(\underline{X}, \underline{x}, \square)$$

where, analogous to (7), $\underline{X} - X = \underline{x} - x = \square$ is an isolated point of X. In the ordinary theory, if $X \supset A \supset B$ then the homology sequence of the triple (X, A, B) is defined to be the sequence

$$\cdots \xrightarrow{\bar{\partial}} H_q(A, B) \xrightarrow{i_*} H_q(X, B) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\bar{\partial}} H_{q-1}(A, B) \xrightarrow{i_*} \cdots$$

where i_* and j_* are induced by inclusion maps and $\bar{\partial}$ is defined (see [1], p. 24) in terms of the ordinary boundary homomorphisms induced by inclusion maps. It is shown in [1], pp. 24—29, that this is an *exact sequence*, and since the proof involves only the first four axioms of homology theory, which are identical with the first four axioms of the reduced homology theory, it follows that the reduced homology sequence of the triple (X, A, B), which is obtained by replacing the groups and homomorphisms above by their reduced theory counterparts, is *exact*. In particular, the homology sequence of the triple $(\underline{X}, x, \square)$ is exact.

Consider the familiar «butterfly» diagram



in which the sequence from lower left to upper right is a portion of the homology sequence of the triple $(\underline{X}, \underline{x}, \underline{\square})$, and the sequence from lower right to upper left is the sequence which is to be proved exact. The homomorphisms $\tilde{i}$, $\tilde{j}$, $\tilde{k}$, and $\tilde{e}$ are induced by inclusions, and $\tilde{\underline{\varphi}}'$ is induced by the restriction $\underline{\varphi}'$ of $\underline{\varphi}$ to $\underline{x}$. Since $\underline{\varphi}'$ is a homeomorphism it is a direct consequence of $\tilde{A}_1$ and $\tilde{A}_2$ that $\tilde{\underline{\varphi}}'$ is an *isomorphism*. In addition, $\tilde{k}$ is an *isomorphism* by Lemma 2. Moreover, commutativity holds in the triangles by $\tilde{A}_2$.

Now let

$$a \varepsilon (\text{Ker } \tilde{\underline{\varphi}}) \cap (\text{Ker } \tilde{j}).$$

Since $a \varepsilon \text{Ker } \tilde{j}$, and since the homology sequence of $(\underline{X}, \underline{x}, \underline{\square})$ is exact, there is an element $b \varepsilon \tilde{H}(\underline{x}, \underline{\square})$ such that $\tilde{i}(b) = a$.

Since $a \varepsilon \text{Ker } \tilde{\underline{\varphi}}$,

$$\tilde{\underline{\varphi}}'(b) = \tilde{\underline{\varphi}}\, \tilde{i}(b) = \tilde{\underline{\varphi}}(a) = 0.$$

But $\tilde{\underline{\varphi}}'$ is an isomorphism, hence $b = 0$ and, therefore, $a = \tilde{i}(b) = 0$. Thus

(19) $$(\text{Ker } \tilde{\underline{\varphi}}) \cap (\text{Ker } \tilde{j}) = 0.$$

Now consider any element $d \varepsilon \tilde{H}_0(\underline{X}, \underline{\square})$, and let $d_1 = \tilde{j}(d)$, $d_2 = \tilde{k}^{-1}(d_1)$, and $g = \tilde{e}(d_2)$. By commutativity one has

$$\tilde{j}(d) = d_1 = \tilde{k}(d_2) = \tilde{j}\, \tilde{e}(d_2) = \tilde{j}(g).$$

Hence $\tilde{j}(d - g) = \tilde{j}(d) - \tilde{j}(g) = 0$, and, therefore,

(20) $$d - g \,\varepsilon\, \text{Ker } \tilde{j}.$$

Now suppose that

$$d \,\varepsilon\, \operatorname{Ker} \underline{\tilde{\varphi}}.$$

Since $g = e(d_2)\,\varepsilon\, \operatorname{Im} \tilde{e}$, it follows from (18) that $g \,\varepsilon\, \operatorname{Ker} \tilde{\varphi}$. Hence

$$d - g \,\varepsilon\, \operatorname{Ker} \underline{\tilde{\varphi}}$$

and $d - g = 0$ by (20) and (19). Consequently, $d = g$ and, therefore,

$$d \,\varepsilon\, \operatorname{Im} \tilde{e}.$$

Thus $\operatorname{Ker} \underline{\tilde{\varphi}} \subset \operatorname{Im} \tilde{e}$, and the lemma has been proved.

The reader is now referred to the remark just before Lemma 3.

Lemma 4. The sequence

$$\tilde{H}_q(X, A) \xrightarrow{\tilde{e}} \tilde{H}_q(\underline{X}, \underline{A}) \xrightarrow{\underline{\tilde{\varphi}}} \tilde{H}_q(\underline{P_0}, \underline{\varphi(A)})$$

is exact for each integer q and all pairs (X, A).

Proof. If $A = \square$ and $q = 0$, Lemma 4 reduces to Lemma 3. In any other case, $\tilde{H}_q(\underline{P_0}, \underline{\varphi(A)}) = \tilde{H}_q(\underline{P_0}, \underline{P_0}) = 0$, and $\tilde{e}$ is an isomorphism by $\tilde{A}_6$. Thus one has

$$\operatorname{Im} e = \tilde{H}_q(\underline{X}, \underline{A}) = \operatorname{Ker} \underline{\tilde{\varphi}}.$$

The sequence in question is therefore exact.

3.4. Note once more the remark preceding Lemma 3, and consider the diagram

$$\begin{array}{ccc}
\tilde{H}_q(\underline{P_0}, \underline{\varphi(A)}) & \xrightarrow{\varepsilon} & \underline{H}_q(P_0, \varphi(A)) \\
\uparrow \underline{\tilde{\varphi}} & & \uparrow \varphi_{\text{III}} \\
\tilde{H}_q(\underline{X}, \underline{A}) & \xrightarrow{\eta} & \underline{H}_q(X, A) \\
\uparrow \tilde{e} & & \uparrow K \\
\underline{\tilde{H}}_q(X, A) & & \underline{\tilde{H}}_q(X, A)
\end{array}$$

in which $\tilde{e}$ is induced by an inclusion map, $\tilde{\varphi}$ is induced by the collapsing map of (1), φ_{III} is defined in (15), while ε and η are the projection isomorphisms of (13). The inclusion homomorphism K maps $\underline{\tilde{H}}_q(X, A)$ identically onto $\operatorname{Ker} \varphi_{\text{III}}$ (see (3) and (1.2)). One has commutativity in the rectangle by the definition of φ_{III} (see (15)), and since ε and η are isomorphisms,

$$\eta\,(\mathrm{Ker}\,\underline{\tilde{\varphi}}) = \mathrm{Ker}\,\varphi_{m} = K(\underline{\tilde{H}}_q(X, A)).$$

Moreover, by Lemma 4 and $\tilde{A}_6$,

$$\tilde{e} \text{ is an isomorphism onto Ker } \underline{\tilde{\varphi}}.$$

Hence

$$\eta\,\tilde{e}\,(\tilde{H}_q(X, A)) = \eta\,(\mathrm{Ker}\,\underline{\tilde{\varphi}}) = K(\underline{\tilde{H}}_q(X, A)).$$

Hence $K^{-1}\eta\,\tilde{e}$ is an isomorphism from $\tilde{H}_q(X, A)$ onto $\underline{\tilde{H}}_q(X, A)$.

Define the family $\underline{\underline{k}}$ of isomorphism $k(q, X, A)$ by

$$k(q, X, A) = K^{-1}\,\eta\,\tilde{e}.$$

With this selection of isomorphisms, the commutativity properties mentioned in (3.2) are readily demonstrable and one has

Theorem **4.** *The family* $\underline{\underline{k}}$ *maaps* $\underline{\underline{\tilde{H}}}$ *isomorphically onto* $R\{S(\tilde{H})\}$.

4. Remarks

4.1. Consider the class $\{\underline{\underline{H}}\}$ of homology theories $\underline{\underline{H}}$. The binary relation of isomorphism between members of this class is an equivalence and thus defines a quotient class $\mathfrak{H}$. The quotient class $\tilde{\mathfrak{H}}$ is obtained in a similar way from the class $\{\underline{\underline{\tilde{H}}}\}$ of reduced homology theories.

It is easy to see that if $\underline{\underline{H}}$ and $\underline{\underline{K}}$ are isomorphic homology theories then $R(\underline{\underline{H}})$ and $R(\underline{\underline{K}})$ are isomorphic reduced homology theories. Hence by Theorem 1 the construction R gives rise to a function

$$\mathfrak{R} : \mathfrak{H} \longrightarrow \tilde{\mathfrak{H}}.$$

Similarly, if $\tilde{H}$ and $\tilde{K}$ are isomorphic reduced theories, application of the construction S yields isomorphic homology theories; thus its follows from Theorem 2 that S induces a function

$$\mathfrak{S} : \tilde{\mathfrak{H}} \longrightarrow \mathfrak{H}.$$

Using this terminology, Theorem 3 states that the composition $\mathfrak{S}\mathfrak{R}$ is the identiy transformation on $\mathfrak{H}$, while Theorem 4 states that $\mathfrak{R}\mathfrak{S}$ is the identity transformation on $\mathfrak{H}$.

Consequently, $\mathfrak{R}$ and $\mathfrak{S}$ are both onto and one-to-one correspondences; hence, to each homology theory corresponds a reduced homology theory which is *unique* up to isomorphism, and conversely, to each reduced homology theory corresponds a homology theory which is *unique* in the same sense.

It is shown in [1], Chapter III, that any two homology theories agree, up to a natural isomorphism, on triangulable pairs. Using these results the corresponding statement is true for reduced homology theories.

4.2. The coefficient group. In any homology theory $\underline{\underline{H}}$ the *coefficient group* is defined to be the group $H_0(P_0)$ where P_0 is the fixed point of (1.1).

This concept can be carried over to a reduced theory $\underline{\underline{\widetilde{H}}}$ by defining the coefficient group G to be $\widetilde{H}_0(P_0 \cup Q_0)$ where P_0 and Q_0 are two fixed points.

One could also have defined the coefficient group in a *non-intrinsic* fashion by stating that it is $\underline{H}_0(P_0)$ where $\underline{H} = S(\widetilde{H})$. The consistency of these two methods of arriving at the coefficient group is readily etablished. By Lemma 1, the homomorphism

$$\widetilde{j} : H_0(P_0 \cup Q_0) \longrightarrow \widetilde{H}_0(P_0 \cup Q_0, Q_0)$$

induced by the inclusion j is an isomorphism. Moreover, $(P_0 \cup Q_0, Q_0) \,\varepsilon\, \mathfrak{F}(P_0, \square)$, hence one has the projection isomorphism

$$\eta : \widetilde{H}_0(P_0 \cup Q_0, Q_0) \longrightarrow \underline{H}(P_0)$$

of (13). Consequently, the composition $\eta\,\widetilde{j}$ is a natural isomorphism from the coefficient group $H_0(P_0 \cup Q_0)$ to the coefficient group $\underline{H}_0(P_0)$. In short, theories which correspond under the transformations $\mathfrak{R}$ and $\mathfrak{S}$ have isomorphic coefficient groups.

4.3. Cohomology theories. A reduced cohomology theory is defined to be a triple of functions $\underline{\underline{\widetilde{H}}} = \widetilde{H}, \sim, \widetilde{\delta}$. The function $\widetilde{H}$ attaches to each pair (X, A) and each integer q an abelian group $\widetilde{H}^q(X, A)$. The function $\sim$ attaches to each map $f : (H, A) \longrightarrow (Y, B)$ and each integer q a homomorphism

$$\widetilde{f}^q : \widetilde{H}^q(Y, B) \longrightarrow \widetilde{H}^q(X, A).$$

The value of the function $\widetilde{\delta}$ is a homomorphism

$$\widetilde{\delta}(q, X, A) : \widetilde{H}^q(X, A) \longrightarrow \widetilde{H}^{q+1}(A).$$

In addition, the functions are required to satisfy seven axioms $\widetilde{A}^1, \ldots, \widetilde{A}^7$ analogous to those for a reduced homology theory. With the exception of $\widetilde{A}^6$ the axioms are merely those of 0.2 with the obvious changes required by the reversed direction of the induced homomorphisms and the boundary homomorphism. In $\widetilde{A}^6$ the concept *isomorphism into* is replaced by its usual dual, *homomorphism onto;* specifically $\widetilde{A}^6$. If U is an open set such that $\bar{U} \subset A^0 \subset A \subset X$, then the homomorphisms $\tilde{e}$ induced by the inclusion map $e:(X-U, A-U) \longrightarrow (X, A)$ is *onto* for each integer q; furthermore, if $U \neq A$ or $q \neq 0$, then $\tilde{e}$ is an *isomorphism.*

The axioms for ordinary cohomology theory are found in [1] and are the duals of those recorded in 3.

Constructions R' (see [1], pp. 18—22) aud S' can be developed corresponding to the construction R and S of 1.1 and 2.3, so that Theorems 1, 2, 3, and 4 have their exact counterpats for cohomology theory and reduced cohomology theory.

4.4. **Continuity**. A property of crucial importance in Cech theories is *continuity* (see [1], Chapter X). It can be shown that continuity is preserved under the constructions R and S for homology theories, and likewise under the constructions R' and S' for cohomology theories. It is not intended to exploit here the full power of this statement, but merely to work with continuous *cohomology* theories.

Let $\underline{\underline{C}}$ stand for the Cech cohomology theory with a coefficient group which is an abelian group G. Let $\widetilde{C} = R(\widetilde{C})$ and call it, as usual, the reduced Cech cohomology theory. Note that by 4.2 the coefficient group of $\underline{\underline{\widetilde{C}}}$ is naturally isomorphic to G.

Remark. Using the above comments and [1], p. 288, one obtains the following result: *If* $\underline{\underline{\widetilde{H}}}$ *is a reduced cohomology theory on compact pairs of spaces, and is continous, then* $\underline{\underline{\widetilde{H}}}$ *is isomorphic to the reduced* Cech *theory with the same coefficient group.*

4.5. **The Radò theory.** In his paper [4] T. Radò defines a system which can be proved to be a reduced cohomology theory. For easy reference the basic definitions of the objects composing the Rado theory are given here.

Let X be a compact space and X^m the m-fold cartesian product of X with itself. If $A \subset X$ a function $f: X^m \longrightarrow G$ (where G is an abelian group) is *zero on* A if $f(A^m) = 0$. A function $f: X^m \longrightarrow G$ is *locally zero on* A if for each $a \varepsilon A$ there is a neighborhood V of such that f is zero on V.

Let $C^q(X, A)$ denote the set of all functions.

$$f: X^{q+2} \longrightarrow G$$

which are locally zero on X and zero on a compact subset A. Using the binary operation of addition of functions $C^q(X, A)$ becomes an abelian group. (For $q < 0$ define $C^q(X, A) = 0$.)

Define the *co-boundary operator* $\delta^q : C^q(X, A) \longrightarrow C^{q+1}(X, A)$ by

$$(\delta^q c)(x_0, \ldots, x_{q+2}) = \sum_{i=0}^{q+2} (-1)^i c(x_0, \ldots, \widehat{x_i}, \ldots, x_{q+2})$$

where $(x, \ldots, \widehat{x_i}, \ldots, x_1)$ denotes that the i-th coordinate is omitted. It is a standard result that the homomorphism defined in this manner satisfies the condition

$$\delta^{q+1} \delta^q = 0.$$

Thus, for any (X, A) the system $\{C^q(X, A), \delta^q\}$ forms a co-chain complex. (See [1], Chapter V.).

If $f : (X, A) \longrightarrow (Y, B)$ is a map the induced homomorphism

$$f''' : C^q(Y, B) \longrightarrow C^q(X, A)$$

is defined by

$$(f''' c)(x_0, \ldots, x_{q+1}) = c(f(x_0), \ldots, f(x_{q+1})).$$

Using the standard techniques for deriving a triple from a chain complex one defines the groups $\widetilde{H}^q(X, A)$, induced homomorphisms $\widetilde{f}$, and a coboundary homomorphism $\widetilde{\delta}$.

The following statements about $\underline{\underline{\widetilde{H}}} = \{\widetilde{H}, \sim, \widetilde{\delta}\}$ are true.

$\underline{\underline{\widetilde{H}}}$ *satisfies* $\widetilde{A}^1, \ldots, \widetilde{A}^4$.

This follows from the method by which one passes from a co-chain complex to the triple $\underline{\underline{\widetilde{H}}}$. (See [1], Chapter V).

$\underline{\underline{\widetilde{H}}}$ *satisfies* $\widetilde{A}^5$.

For a proof see [5], pp. 87—79. A different, somewhat simpler proof can be given which deals with $\widetilde{A}^5$ at the co-chain level.

$\underline{\underline{\widetilde{H}}}$ *satisfies* $\widetilde{A}^6$.

For a proof, see [5], p.p. 70—73.

$\underline{\underline{\widetilde{H}}}$ *satisfies* $\widetilde{A}^7$.

This follows from the fact that $C^q(P)$ is trivial for all q.

Hence $\underline{\underline{\widetilde{H}}}$ *is a reduced cohomology theory.*

In addition, it follows from [5], pp. 80—86, that $\underset{=}{\tilde{H}}$ is *continuous*. Therefore, in consequence of 4.4, *the theory* $\underset{=}{\tilde{H}}$ *is isomorphic to the reduced* ČECH *theory on compact pairs.*

BIBLIOGRAPHY

[1] S. EILENBERG AND N. STEENROD : **Foundations of Algebraic Topology**, Princeton, (1952).

[2] » : *Axiomatic Approach to Homology Theory*, Proc. Nat. Acad. Sciences, U. S. A., 31, pp. 117—120, (1945).

[3] S. T. HU : *Axiomatic Approach to the Homotopy Groups*, Bul. Amer. Math. Soc., 62, No. 5, pp. 490—504, (1956).

[4] T. RADO : *On General Cohomology Theory*, Proc. Amer. Math. Soc., 4, pp. 244—246 (1953).

[5] T. RADO AND P. V. REICHELDERFER : **Continuous Transformations in Analysis**, Springer, Berlin, (1955).

GRADUATE INSTITUTE
FOR
MATHEMATICS AND MECHANICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA, U.S.A.

(Manuscript received May 12, 1960)

ÖZET

Bu makalenin gayesi indirgenmiş homoloji (ve kohomoloji) teorisine aksiyomatik bir yol temin etmektir. İndirgenmiş homoloji teorisi, EILENBERG ve STEENROD'un homoloji'ye ait malûm aksiyomlarının benzeri olan yedi tane aksiyom sağlayan $(\tilde{H}, \sim\tilde{\partial})$ gibi bir üçlü olarak tarif edilmektedir. Bir H homoloji teorisi verildiğine göre, EILENBERG ve STEENROD'un mânasında bir $R(H)$ indirgenmiş homoloji teorisi meydana getiren bir R konstrüksiyonu bu müellifler tarafından elde edilmiştir. $R(H)$'ıa, bu makalede verilen indirgenmiş homoloji teorisine ait aksiyomları sağladığı gösterilmektedir. Bundan başka, bir homoloji teorisi meydana getirmek üzere, herhangi indirgenmiş bir homoloji teorisine tatbik edilebilen ikinci bir S konstrüksiyonu verilmektedir. S ve R konstrüksiyonlarının uygun bir mânada biribirinin tersi oldukları gösterilerek indirgenmiş teorinin karakterizasyonu tamamlanmıştır. Gerçekten, bu konstrüksiyonlar izomorf homoloji teorileri sınıfları ile izomorf indirgenmiş homoloji teorileri sınıfları arasında birebir bir tekabül temin ederler; böylece bu iki tip sistemden herhangi biri, topolojik uzayların cebrik etüdü için bir hareket noktası olarak alınabilir. Tatbikat olmak üzere gösterilebilir ki, bir homoloji teorisi süreklilik aksiyomunu sağlarsa, mütekabil indirgenmiş teori de bu aksiyomu sağlar, ve bunun aksi de doğrudur. Böylece süreklilik aksiyomu, indirgenmiş ČECH homoloji teorisini karakterize etmeye yarar.

Yukarıdakilere benzer ihtarlar kohomoloji teorisi için câridir.