PITCH OF A RECTILINEAR CONGRUENCE

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In this paper the author has found some expressions for the pitch of a general rectilinear congruence and for an isotropic one.
For a rectilinear congruence the expression for the pitch of a ray takes the form

$$p = \int\int e^{\gamma\delta} \mu_{\gamma\delta} \frac{dS}{\sqrt{g}},$$

while for an isotropic congruence the pitch is

$$p = 2 \int\int \Phi \frac{dS}{\sqrt{g}}.$$

1. The *pitch* of a ray of a rectilinear congruence has been studied by RAM BEHARI [1]. Let a rectilinear congruence be defined by the coordinates x^i $(i = 1, 2, 3)$ of a point M and λ^i $(i = 1, 2, 3)$ be the direction cosines of the ray l passing through M, where x^i and λ^i are functions of two parameters u^α $(\alpha = 1, 2)$. When M moves along a closed eurve C these rays generate a ruled surface. An orthogonal trajectory is drawn to this ruled surface which intersects the ray l passing through the point M at two points P_1 and P_2: this distance $P_1 P_2$ is called the pitch of the pencil of rays of the congruence.

The object of this paper is to find some expressions for the pitch of a pencil of rays of a rectilinear congruence and that of an isotropic congruence.

The expression for the pitch is

$$p = \int_C \lambda^i \, dx^i,$$

which by means of GREEN's formula takes the form

$$p = \int\int_\Sigma (\lambda^i{}_1 \, x^i{}_2 - \lambda^i{}_2 \, x^i{}_1) \, du^1 \, du^2, \tag{1.1}$$

where the suffixes denote the differentiation with respect to the parameters u^1 and u^2 based on the fundamental tensor of the surface of reference

$$g_{\alpha\beta} = x^i{}_{,\alpha} \cdot x^i{}_{,\beta}$$

and the double integral is extended to the area Σ of the portion of the surface of reference $x^i = x^i(u^1, u^2)$ bounded by C.

We now write (1.1) as

$$(1.2) \qquad \begin{aligned} p &= \iint (\mu_{12} - \mu_{21}) \frac{\sqrt{G_{\alpha\beta}\, du^\alpha\, du^\beta}}{\sqrt{g}} \\ &= \iint e^{\gamma\delta} \mu_{\gamma\delta} \frac{\sqrt{G_{\alpha\beta}\, du^\alpha\, du^\beta}}{\sqrt{g}}, \end{aligned}$$

which can be written in the alternative form as

$$(1.3) \qquad = \iint e^{\gamma\delta} \mu_{\gamma\delta} \frac{dS}{\sqrt{g}}$$

where dS is the corresponding element of area on the spherical representation of the congruence bounded by the parametric curves, integration now being extended to that portion of the unit sphere corresponding to Σ, $e^{\gamma\delta}$ is defined by $e^{11} = e^{22} = 0$ and $e^{12} = 1$, $e^{21} = -1$,

$$g = |g_{\alpha\beta}| \quad \text{and} \quad \mu_{\alpha\beta} = \lambda^i{}_{,\alpha} \cdot x^i{}_{,\beta}.$$

If on a sheet of the focal surface of the congruence a family of curves be taken as the parametric curves $u^2 = \text{constant}$, and their orthogonal trajectories as the parametric curves $u^1 = \text{constant}$, we get $x^i{}_2 = 0$ since in this case the coordinates become functions of only one parameter u^1. The expression (1.3) reduces to

$$(1.4) \qquad p = - \iint \mu_{21} \frac{dS}{\sqrt{g}}.$$

Also, along the curve $u^2 = \text{constant}$ on the surface of reference

$$\lambda^i = x^{i\prime} = x^i{}_1\, u^{1\prime} = x^i{}_1 \frac{1}{\sqrt{g_{11}}},$$

since in this case $(dS)^2 = g_{11} (du^1)^2$. Dashes denote differentiation with respect to the arc-length s of the curve on the surface of reference.

Therefore

$$\lambda^i{}_2 = - x^i{}_1 \left\{ \frac{1}{2} \frac{1}{(g_{11})^{3/2}} \right\} \frac{\partial}{\partial u^2} (g_{11}).$$

Now, in terms of the fundamental magnitudes of the surface of reference, the pitch of a pencil of rays of a rectilinear congruence assumes the form

(1.5)
$$p=\iint (x^i{}_1)^2 \frac{1}{2\,(g_{11})^{3/2}} \frac{\partial}{\partial u^2}(g_{11}) \frac{dS}{\sqrt{g}}$$
$$=\iint \frac{1}{2\,(g_{11})^{1/2}} \frac{\partial}{\partial u^2}(g_{11}) \frac{dS}{\sqrt{g}}.$$

In case the congruence is formed by the *normals* to the surface of reference the orthogonal trajectory will become a closed curve. Thus, obviously, the pitch vanishes for a peneil of rays of a normal congruence.

From (1.2) we observe that the pitch of a pencil of rays of a rectilinear congruence vanishes when

$$\mu_{\alpha\beta}=\mu_{\beta\alpha},$$

i.e. the pitch of a pencil of rays of a normal congruence vanishes.

Hence the well known result:

The necessary and sufficient condition that the congruence be normal is that the pitch vanishes for every pencil of rays of the congruence.

If in the equation (1.5) g_{11} is constant the pitch vanishes and vice-versa, since the surface area included by the parametric curves $u^1=$ constant and $u^2=$ constant and $a^1+da^1=$ constant and $u^2+da^2=$ constant is not zero.

Thus we conclude that:

The necessary and sufficient condition that the pitch of a pencil of rays of a rectilinear congruence vanishes is that the fundamental magnitudes of the surface of reference be constant.

2. Pitch of an isotropic congruence.

For an isotropic congruence we have [2]

(2.1)
$$\mu_{\alpha\beta}=\Phi\, E_{\alpha\beta}$$

where Φ is BIANCHI's characteristic function, and $E_{\alpha\beta}=(\lambda^i\, \lambda^i{}_{,\alpha}\, \lambda^i{}_{,\beta})$.

By virtue of (2.1) the formula (1.2) reduces to the form

(2.2)
$$p=\iint \Phi\, e^{\gamma\delta}\, E_{\gamma\delta} \frac{dS}{\sqrt{g}}$$
$$=2\iint \Phi \frac{dS}{\sqrt{g}}.$$

It has been shown by MISHRA in the paper quoted above that for an isotropic congruence BIANCHI's characteristic function is the factor of proportionality between SANNIA's quadratic forms, and this factor can never be a constant other than zero, hence in this case the pitch vanishes. Thus, we conclude that:

The pitch of a pencil of rays of an isotropic congruence vanishes when Φ is constant.

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REFERENCES.

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[2] R. S. MISHRA : *On isotropic congruences*, Ganita, 2, Pp. 45–49, (1951).

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ÖZET

Bu araştırmada müellif doğru kongrüansları için RAM BEHARI [1] tarafından ithal edilip «pitch» adı verilen invaryantı hesaplamıştır.

Umumî bir doğru kongrüansı halinde

$$p = \iint e^{\gamma\delta} \mu_{\gamma\delta} \frac{dS}{\sqrt{g}}$$

şeklinde ifade edilen «pitch», kongrüans izotrop olduğu takdirde

$$p = 2 \iint \Phi \frac{dS}{\sqrt{g}}$$

şekline girer.