Φ - CONGRUENCES

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In this paper the author has studied some properties of the principal surfaces of the «Φ-congruences» defined by Upadhyay [8 and 9]. The equations of the principal surfaces of the «Φ-congruences» have been obtained in a determinant form. Besides this the following properties have been obtained:

(1) The necessary and sufficient condition that the lines of the Φ-congruence be parallel is that the skewnesss of distribution of the congruence-λ be equal to the cotangent of the constant angle Φ.

(2) If the spherical representations of the Φ-congruonce are minimal lines, its lines are parallel.

(3) The line of striction of the Φ-congruence will lie on its surface of reference if any one of the following relations hold:

(i) The Φ-congruence is parallel to the congruence-λ,

(ii) The line of striction of the congruence-λ lies on its surface of reference,

(iii) The lines of the congruence-λ are parallel to a plane, provided that the lines of the Φ-congruence are not parallel or the spherical representation of its ruled surfaces are not minimal lines.

(4) The principal planes of the Φ-congruence in general and the surfaces corresponding to one of tho parametric curves of the surface of reference are inclined at a constant angle.

1. Φ-congruences have been defined and studied by Upadhyay [8] and [9]. Let x^i $(i = 1, 2, 3)$ be the coordinates of a point M on the surface of reference and λ^i $(i = 1, 2, 3)$ be the direction cosines of a line of a congruence passing through M and let this congruence be called the original congruence. Let λ^{*i} be the direction cosines of a ray of another congruence intersecting the consecutive rays of the original congruence under a constant angle Φ: this congruence is then called a Φ-congruence. The lines of striction of the ruled surfaces of the original congruence lie on a surface, which will be assumed to be fixed. We shall denote this surface by S^* throughout and the original congruence will also be called the congruence-λ. The surface S^* is taken as the surface of reference of the Φ-congruence.

The object of this paper is to find the expressions for the principal surfaces of the Φ-congruence and their properties. Some particular cases have also been considered.

2. Let a ray of the Φ-congruence with direction cosines λ^{*i} intersect its surface of reference at a point P, whose coordinates are y^i and such that

$$y^i = x^i + t\,\lambda^i \tag{2.1}$$

where t is the distance of the central point of the ruled surface of the congruence-λ from the point M and x^i, y^i, λ^i and λ^{*i} are all functions of u^α $(\alpha = 1, 2)$ [1]. Farther

$$\lambda^{*i} \cdot \lambda^{*i} = 1. \tag{2.2}$$

For convenience the notation $\lambda^{*i}{}_{,\alpha}$ for the covariant derivative of λ^{*i} with respect to first fundamental tensor $G^*{}_{\alpha\beta}$ of the spherical representation of the Φ-congruence is used instead of $\partial\lambda^{*i}/\partial u^\alpha$ so that the two quadratic forms used by KUMMER [3] are

$$G^*{}_{\alpha\beta}\, du^\alpha\, du^\beta \tag{2.3}$$

and

$$\mu^*{}_{\alpha\beta}\, du^\alpha\, du^\beta \tag{2.4}$$

where

$$G^*{}_{\alpha\beta} = \lambda^{*i} \cdot \lambda^{*i}{}_{,\beta} \tag{2.5}$$

and

$$\mu^*{}_{\alpha\beta} = (y^i{}_{,\beta} \cdot \lambda^{*i}{}_{,\alpha}). \tag{2.6}$$

SANNIA's two quadratic forms [1] as modified by MISHRA [4] are then

$$G_{\alpha\beta}\, du^\alpha\, du^\beta \tag{2.7}$$

and

$$\xi^*{}_{\alpha\beta}\, du^\alpha\, du^\beta \tag{2.8}$$

where

$$\xi^*{}_{\alpha\beta} = (y^i{}_{,\alpha}, \lambda^{*i}, \lambda^{*i}{}_{,\beta}). \tag{2.9}$$

It may be noted that similar quantities without asterisks correspond to the congruence-λ. λ^{*i} can be expressed as [6]

$$\lambda^{*i} = \lambda^i \cos\Phi + \lambda^i \times \frac{d\lambda^i}{d\sigma} \sin\Phi \tag{2.10}$$

where $d\sigma$ is the linear differential element of length of the spherical representation of the congruence-λ.

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1) In what follows Latin indices take values (1, 2, 3) and Greek indices the values (1, 2).

The function λ^{*i} may be expressed in terms of the direction numbers $y^i{}_{,\alpha}$ of the tangents to the coordinate curves of the surface S^* through P and the direction cosines X^i of the normal to the surface S^* at P. Thus

$$\lambda^{*i} = p^{*\alpha} y^i{}_{,\alpha} + q^* x^i \tag{2.11}$$

where $p^{*\alpha}$ are contravariant components of a unit vector on the surface S^* at P, q^* is a positive scalar function and $y^i{}_{,\alpha}$ denotes covariant differentiation of y^i with respect to u^α based on the fundamental tensor $G^*{}_{\alpha\beta}$.

From (2.10) we get [9]

$$\lambda^{*i}{}_{,\alpha} = \lambda^i{}_{,\alpha} \cos\Phi + \lambda^i{}_{,\alpha} \times \lambda^i{}_{,\gamma}\, u'^\gamma \sin\Phi + \lambda^i \times \lambda^i{}_{,\gamma}\, u'^\gamma{}_{,\alpha} \sin\Phi. \tag{2.12}$$

3. **Principal surfaces of a Φ-congruence.**

The distance of the central point of a line of the Φ-congruence from its surface of reference is given by [10]

$$\begin{aligned} t^* &= -\left(\frac{dy^i}{d\sigma^*} \cdot \frac{d\lambda^{*i}}{d\sigma^*}\right) \\ &= -\left(\frac{dy^i \cdot d\lambda^{*i}}{d\sigma^{*2}}\right) \\ &= -\frac{(y^i{}_{,\alpha} \cdot \lambda^{*i}{}_{,\beta} + \lambda^{*i}{}_{,\alpha} \cdot y^i{}_{,\beta})}{2G^*{}_{\alpha\beta}\, du^\alpha\, du^\beta} \\ &= -(\mu^*{}_{\alpha\beta} + \mu^*{}_{\beta\alpha})\, du^\alpha\, du^\beta / 2G^*{}_{\alpha\beta}\, du^\alpha\, du^\beta \end{aligned} \tag{3.1}$$

where [8]

$$\begin{aligned} \mu^*{}_{\alpha\beta} &= \mu_{\alpha\beta} \cos\Phi + (p_\beta E_{\alpha\gamma} u'^\gamma + \xi_{\beta\gamma} u'^\gamma{}_{,\alpha} \\ &\qquad + t E_{\gamma\beta} u'^\gamma{}_{,\alpha}) \sin\Phi + t\, G_{\alpha\beta} \cos\Phi \\ G^*{}_{\alpha\beta} &= G_{\alpha\beta} \cos^2\Phi + (E_{\gamma\beta} u'^\gamma{}_{,\alpha} - E_{\alpha\delta} u'^\delta{}_{,\beta}) \sin\Phi \cos\Phi \\ &\qquad + (E_{\alpha\gamma} E_{\beta\delta} u'^\gamma u'^\delta + G_{\gamma\delta} u'^\gamma{}_{,\alpha} u'^\delta{}_{\beta}) \sin^2\Phi. \end{aligned} \tag{3.2}$$

The equation of the principal surfaces of the Φ-congruence are obtained by equating to zero the derivative of t^* in (3.1) with respect to du^α/du^β ($\alpha \neq \beta$). That is

$$\begin{aligned} &2G^*{}_{\gamma\delta}\, du^\gamma\, du^\delta (\mu^*{}_{\alpha\beta} + \mu^*{}_{\beta\alpha})\, du^\alpha \\ &- 2(\mu^*{}_{\gamma\delta} + \mu^*{}_{\delta\gamma})\, du^\gamma\, du^\delta\, G^*{}_{\alpha\beta}\, du^\alpha = 0 \end{aligned} \tag{3.3}$$

or

$$(\mu^*{}_{\alpha\beta} + \mu^*{}_{\beta\alpha})\, du^\alpha + 2t^* G^*{}_{\alpha\beta}\, du^\alpha = 0.$$

Eliminating t^* we get

$$\left| \begin{matrix} (\mu^*_{1\alpha} + \mu^*_{\alpha 1})\, du^\alpha & (\mu^*_{2\alpha} + \mu^*_{\alpha 2})\, du^\alpha \\ G^*_{1\alpha}\, du^\alpha & G^*_{2\alpha}\, du^\alpha \end{matrix} \right| = 0. \tag{3.4}$$

From (3.3) we obtain the value of t^* as

$$|\, (\mu^*_{\alpha\beta} + \mu^*_{\beta\alpha}) + t^*\, G^*_{\alpha\beta} \,| = 0 \tag{3.5}$$

which on expansion gives

$$\begin{aligned} t^* (G^*_{11}\, G^*_{22} - G^*_{12}) + t^* \Big\{ G^*_{11}\, \mu^*_{22} + \mu^*_{11}\, G^*_{22} \\ - G^*_{12}\, (\mu^*_{12} + \mu^*_{21}) \Big\} - \Big\{ \mu^*_{12}\, \mu^*_{21} \\ - \frac{(\mu^*_{12} + \mu^*_{21})}{4} \Big\} = 0. \end{aligned} \tag{3.6}$$

Equations (3.4) and (3.6) are of the same form as those obtained by Weatherburn [10].

If we choose the parameters in such a way that the principal surfaces correspond to parametric curves, assuming moreover that Kummer's two quadratic forms are not proportional, from (3.4) we get

$$G^*_{12} = 0, \quad \mu^*_{12} + \mu^*_{21} = 0 \tag{3.7}$$

With this choice of the parameters the distance of the central point of a line of the Φ-congruence from its surface of reference is given by

$$t^* = - \frac{\mu^*_{\alpha\beta}\, du^\alpha\, du^\beta}{G^*_{\alpha\beta}\, du^\alpha\, du^\beta} \qquad (\alpha = \beta) \tag{3.8}$$

$$\begin{aligned} = - \Big[\Big\{ (\mu_{11} + t\, G_{11}) \cos \Phi + (p_1\, E_{1\gamma}\, u'^\gamma + \xi_{1\gamma}\, u'^\gamma{}_{,1} \\ + t\, E_{\gamma 1}\, u'^\gamma{}_{,1}) \sin \Phi \Big\} (du^1)^2 + \Big\{ (\mu_{22} + t\, G_{22}) \cos \Phi \\ + (p_2\, E_{2\gamma}\, u'^\gamma + \xi_{2\gamma}\, u'^\gamma{}_{,2} + t\, E_{\gamma 2}\, u'^\gamma{}_{,2}) \sin \Phi \Big\} (du^2)^2 \Big] \Big/ \Big[\Big\{ G_{11} \cos^2 \Phi \\ + 2 E_{\gamma 1}\, u'^\gamma{}_{,1} \sin \Phi \cos \Phi + (E_{1\gamma}\, E_{1\delta}\, u'^\gamma\, u'^\delta + G_{\gamma\delta}\, u'^\gamma{}_{,1}\, u'^\delta{}_{,1}) \sin^2 \Phi \Big\} (du^1)^2 \\ + \Big\{ G_{22} \cos^2 \Phi + 2 E_{\gamma 2}\, u'^\gamma{}_{,2} \sin \Phi \cos \Phi + (E_{2\gamma}\, E_{2\delta}\, u'^\gamma\, u'^\delta \\ + G_{\gamma\delta}\, u'^\gamma{}_{,2}\, u'^\delta{}_{,2}) \sin^2 \Phi \Big\} (du^2)^2 \Big]. \end{aligned} \tag{3.9}$$

Dividing both numerator and denominator of (3.9) by $d\sigma^2$ this equation reduces to

$$t^* = -\frac{[\xi_{1\gamma}\,\varrho^\gamma\,(u^1)' + \xi_{2\gamma}\,\varrho^\gamma\,(u^2)' - 2\mu t]\sin\Phi}{2\,(\cos^2\Phi - 2\mu\sin\Phi\cos\Phi + \mu^2\sin^2\Phi)}. \tag{3.10}$$

Now [3]

$$\varrho^\gamma E_{\alpha\gamma}\,u'^\alpha = k_g,$$
$$u'^\gamma{}_{,\beta}\,u'^\beta = \varrho^\gamma \tag{3.11}$$

and [8]

$$G_{\gamma\delta}\,\varrho^\gamma\,\varrho^\delta = k^2{}_g$$

$$E_{\alpha\beta}\,u'^\alpha\,a'^\beta = 0$$

and again

$$\mu_{\alpha\beta}\,u'^\alpha\,u'^\beta = -t$$

$$G_{\alpha\beta}\,u'^\alpha\,u'^\beta = 1$$

where

$$\varrho^\gamma = \frac{d^2u^\gamma}{d\sigma^2} + \left\{\begin{matrix}\gamma\\ \alpha\,\beta\end{matrix}\right\}\frac{du^\alpha}{d\sigma}\,\frac{du^\beta}{d\sigma}$$

is the curvature vector of the spherical representation of the congruence-λ and k_g is the geodesic curvature of the spherical indicatrix of the congruence-λ which is also called the skewness of distribution μ of the congruence-λ [5].

Now since [2] $\varrho^\gamma = \mu\mu^\gamma$, μ^γ being a unit vector in the direction of the curvature of the spherical representation of the congruence-λ

$$t^* = -\frac{\mu\,\{\xi_{1\gamma}\,\mu^\gamma\,(u^1)' + \xi_{2\gamma}\,u^\gamma\,(u^2)' - 2t\}\sin\Phi}{2\,(\cos\varphi - \mu\sin\Phi)^2} \tag{3.12}$$

then if

$$\mu = \cot\Phi$$

we get

$$t^* = \infty. \tag{3.13}$$

Hence the result: *The necessary and sufficient condition that the rays of the Φ-congruence be parallel is that the skewness of distribution of the congruence-λ be equal to the cotangent of the constant angle Φ.*

But [8] if the spherical representation of the Φ-congruence are minimal lines we get $\mu = \cot\Phi$. Hence: *If the spherical representation of the ruled surfaces of Φ-congruence are minimal lines, its lines are parallel.*

In particular,

(1) If $\Phi = 0$ we get

$$t^* = 0 \tag{3.14}$$

(ii) When $$\Phi = \pi/2$$

$$t^* = \frac{\xi_{1\gamma}\,\mu^\gamma\,(u^1)' + \xi_{2\gamma}\,\mu^\gamma\,(u^2)' - 2t}{2\mu}. \tag{3.15}$$

This is an expression for the distance of the central point of the Φ-congruence from its surface of reference when the Φ-congruence is formed by lines at right angles to the rays of the congruence-λ.

Further, if $\mu = 0$, the lines of the congruence are parallel to a plane and

$$t^* = \infty. \tag{3.16}$$

Hence: *The necessary and sufficient condition that the congruence formed by lines at right angles to the rays of the congraence-λ, be parallel, is that the congruence-λ be parallel to a plane.*

(iii) When the congruence-λ is isotropic, we have

$$\xi_{\alpha\beta} = \chi\, G_{\alpha\beta}$$

where χ is the proportionality factor between the coefficients of SANNIA's quadratic forms. We then have

$$t^* = -\frac{\{\chi\,(G_{1\gamma}\,\varrho^\gamma\,(u^1)' + G_{2\gamma}\,\varrho^\gamma\,(u^2)' - 2\mu\, t\}\sin\Phi}{2\,(\cos\Phi - \mu\sin\Phi)^2} \tag{3.17}$$

$$= \frac{\mu\, t}{(\cos\Phi - \mu\sin\Phi)^2} \tag{3.18}$$

since [2]

$$G_{\sigma\beta}\,\varrho^\beta\,u^{1\alpha} = 0.$$

From (3.13) and (3.18) we observe that the line of striction of the Φ-congruence will lie on its surface of reference if one any of the following relations hold.

(i) *The Φ-congruence is parallel to the congraence-λ,*

(ii) *The line of striction of the congraence-λ lies on its surface of reference,*

(iii) *The lines of the congraence-λ are parallel to a plane, provided the lines of the Φ-congruence are not parallel or spherical representations of its ruled surfaces are not minimal lines.*

(i) If $$\Phi = 0$$

from (3.18) we get

$$t^* = 0 \tag{3.19}$$

(ii) when $\Phi = \pi/2$

$$t^* = \frac{t}{\mu} \tag{3.20}$$

which is the result already obtained by Upadhyay [3]. The limits correspond to the parametric curves on the surface of reference: then let the corresponding values of t^* be denoted by t^*_1 and t^*_2 so that

$$\begin{aligned} t^*_1 &= -\frac{\mu^*_{11}}{G^*_{11}} \\ &= -\{(\mu_{11} + t\,G_{11})\cos\Phi + (p_1 E_{1\gamma} u^{1\gamma} \\ &+ \xi_{1\gamma} u^{1\gamma}{}_{,1} + t\,E_{\gamma 1} u^{1\gamma}{}_{,1})\sin\Phi\} / \{G_{11}\cos^2\Phi \\ &+ 2E_{\gamma 1} u^{1\gamma}{}_{,1}\sin\Phi\cos\Phi + (E_{1\gamma}E_{1\delta} u^{1\delta} u'^{\gamma} + G_{\gamma\delta} u^{1\gamma}{}_{,1} u^{1\delta}{}_{,1})\sin^2\Phi\}. \end{aligned} \tag{3.21}$$

Multiplying the numerator and denominator of (3.21) by $\{(u^1)'\}^2$, this formula will then by virtue of (3.11), reduce to

$$\begin{aligned} t^*_1 &= -\frac{(\xi_{1\gamma}\varrho^{\gamma}(u^1)' - \mu t)\sin\Phi}{(\cos\varphi - \mu\sin\Phi)^2} \\ &= -\frac{\mu(\xi_{1\gamma}\mu^{\gamma}(u^1)' - t)}{(\cos\Phi - \mu\sin\Phi)^2}. \end{aligned} \tag{3.22}$$

Similarly

$$\begin{aligned} t^*_2 &= -\frac{\mu^*_{22}}{G^*_{22}} \\ &= -\frac{\mu(\xi_{2\gamma}\mu^{\gamma}(u^2)' - t)}{(\cos\Phi - \mu\sin\Phi)^2}. \end{aligned} \tag{3.23}$$

In particular

(i) When $\Phi = 0$ we get

$$t^*_1 = 0, \qquad t^*_2 = 0. \tag{3.24}$$

(ii) Also when $\Phi = \pi/2$

$$\begin{aligned} t^*_1 &= -\frac{\xi_{1\gamma}\mu^{\gamma} - t}{\mu} \\ t^*_2 &= -\frac{\xi_{2\gamma}\mu^{\gamma} - t}{\mu}. \end{aligned} \tag{3.25}$$

If ϑ be the angle between the common perpendicular of two consecutive rays of the Φ-congruence in the general case and the common perpendicular of two consecutive rays of the congruence corresponding to its principal surface $du^2 = 0$ we have [10]

$$\cos^2 \vartheta = \frac{G^*_{11} (du^1)^2}{G^*_{11} (dn^1)^2 + G^*_{22} (dn^2)^2}$$

(3.26)

$$= \{ G_{11} \cos^2 \Phi + 2E_{\gamma 1} u^{1\gamma}{}_{,1} \sin \Phi \cos \Phi + (E_{1\gamma} E_{1\delta} u^{1\gamma} u^{1\delta}$$

$$+ G_{\gamma\delta} u^{1\gamma}{}_{,1} u^{1\delta}{}_{,1}) \sin^2 \Phi \} (du^1)^2 / \{G_{11} \cos^2 \Phi + 2E_{\gamma 1} n^{\gamma}{}_{,1} \sin \Phi \cos \Phi$$

$$+ (E_{1\gamma} E_{1\delta} u^{1\gamma} u^{1\delta} + G_{\gamma\delta} u^{1\gamma}{}_{,1} u^{1\delta}{}_{,1}) \sin^2 \Phi \} (du^1)^2$$

$$+ \{ G_{22} \cos^2 \Phi + 2E_{\gamma 2} u^{1\gamma}{}_{,2} \sin \Phi \cos \Phi + (E_{2\gamma} E_{2\delta} u^{1\gamma} u^{1\delta}$$

$$+ G_{\gamma\delta} u^{1\gamma}{}_{,2} u^{1\delta}{}_{,2}) \sin^2 \Phi \} (du^2)^2$$

which by virtue of (3.11) reduces to

(3.27)

$$\cos^2 \vartheta = 1/2$$

$$\vartheta = \pi/4 \quad \text{or} \quad 3\pi/4.$$

Therefore ϑ is independent of Φ-

Thus: *The principal planes of the general Φ-congruence and the surfaces corresponding to one of the parametric curves of the surface of reference are inclined at a constant angle.* From equations (3.10), (3.22) and (3.23) the expression

$$t^*_1 \cos^2 \vartheta + t^*_2 \sin^2 \Phi$$

takes the form

(3.28)

$$-\frac{\{ \xi_{1\gamma} \varrho^{\gamma} (u^1)' + \xi_{2\gamma} \varrho^{\gamma} (u^2)' - 2\mu t \} \sin \Phi}{2 (\cos \Phi - \mu \sin \Phi)^2} = t^*$$

which is HAMILTON's formula for Φ-congruences; then when

(3.29)

$$t^* = 0$$

$$\tan \vartheta = \pm \sqrt{-\frac{t^*}{t^*_2}}\ .$$

From (3.28) we get

(3.30)

$$t^*_1 \cos^2 \vartheta + t^*_2 \sin^2 \vartheta$$

$$= -\frac{\mu \{ \xi_{1\gamma} \mu^{\gamma} (u^1)' + \xi_{2\gamma} \mu^{\gamma} (u^2)' - 2t \} \sin \Phi}{2 (\cos \Phi - \mu \sin \Phi)^2}$$

since

$$\varrho^{\gamma} = \mu \mu^{\gamma}.$$

In case

$$\mu = 0$$

(3.31)
$$\tan \vartheta = \pm \sqrt{-\frac{t^*_1}{t^*_2}}.$$

From (3.29) and (3.31) we conclude that: *If the line of striction of the Φ-congraence lies on its surface of reference or the rays of the congruence-λ are parallel to a plane, the inclination of the principal planes of the Φ-congruence in general to the surfaces corresponding to one of the parametric curves on the surface of reference is independent of the angle Φ and is given by the relation* (3.29) *or* (3.31).

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ÖZET

Bu araştırmada müellif, UPADHYAY tarafından [8], [9] tarif edilen «φ-kongrüans» larının esas yüzeylerinin bazı özelliklerini incelemiş ve bu yüzeylerin denklemlerini determinantlar şeklinde ifade etmiştir.

Bundan maada aşağıdaki neticeler elde edilmiştir :

(1) φ-kongrüansını teşkil eden doğruların paralel olmaları için gerek ve yeter şart, λ-kongrüansının tevzi sapmasının sabit φ açısının kotangenline eşit kalmasıdır.

(2) φ-kongrüansının küresel göstergeleri minimal çizgiler ise, kongrüansın doğruları paraleldir.

(3) Aşağıdaki 3 özelikten herhangi birisi tahakkuk etliği takdirde φ-kongrüansının boğaz çizgisi bu kongrüansı tarife yarıyan yüzey üzerindedir.

(ı) φ-kongrüansı λ-kongrüansına paraleldir ;

(ıı) λ-kongrüansının boğaz çizgisi kongrüansın tarifine yarıyan yüzey üzerindedir ;

(ııı) λ-kongrüansının doğruları bir düzleme paralel fakat φ-kongrüansının doğruları paralel değil ve küresel göstergeleri minimal çizgiler değildir.

(4) Umumî φ-kongrüansının esas düzlemleriyle doğrultman yüzey üzerindeki parametrik çizgilere tekabül eden regle yüzeyler sabit açılar altında kesişirler.