ON FRENET'S FORMULAE AND CURVATURES IN A GENERALISED RIEMANN SPACE

A. C. SHAMIHOKE (*)

Abstract. WEATHERBURN [1] had obtained FRENET's formulae in an n-dimensional RIEMANN space which were later extended to generalised RIEMANN space [2] by SAXENA and RAM BEHARI. In this paper another form of FRENET's formulae for a generalised RIEMANN space has been obtained. The equivalence of the two forms has also been established. LEVY [4] had expressed the derivative of the angle between two consecutive binormals of a curve lying in a RIEMANN space with respect to the arc length in terms of the curvatures. This result has also been extended to generalised RIEMANN spaces.

1. FRENET's formulae

Let F_n denote an n-dimensional generalised RIEMANN space endowed with a local coordinate system x^i $(i = 1, \ldots, n)$. To each point $P(x^i)$ of F_n is associated a non-symmetric tensor $g_{ij}(x)$ called the metric tensor. EISENHART [3] has obtained an affine connection Δ^i_{hk} given by

$$\Delta^i_{hk} = \frac{1}{2} g^{\underline{im}} \left(\frac{\partial g_{hm}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^h} - \frac{\partial g_{hk}}{\partial x^m} \right), \tag{1.1}$$

$g^{\underline{ij}}$ being the conjugate tensor of the symmetric part of the metric tensor g_{ij}. In what follows bar and hook will be used to denote the symmetric and skew-symmetric parts of a quantity: thus

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji})$$

and

$$g_{\underset{\smile}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}).$$

Mostly, the notations of EISENHART [3] will be followed.

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We now consider a curve $C: x^i = x^i(s)$ referred to its arc length s as the parameter, lying in a generalised RIEMANN space F_n, so that its unit tangent vector is given by

(1.2) $$t^i \equiv \frac{dx^i}{ds}.$$

We define a system of n vectors $\xi^i{}_{(p)}$ as follows:

(1.3) $$\xi^i{}_{(1)} = t^i, \quad \xi^i{}_{(p)} = \frac{D}{Ds}(\xi^i{}_{(p-1)})$$

D/Ds denoting the intrinsic derivative [3] in the direction of C. To ortogonalize these vectors we define a set of n vectors as follows:

(1.4) $$\eta^i{}_{(1)} = \xi^i{}_{(1)}, \quad \eta^i{}_{(p)} = \frac{1}{\sqrt{D_{p-1}D_p}} \begin{vmatrix} (1,1) & \ldots & (1,p-1) & \xi^i{}_{(1)} \\ (2,1) & \ldots & (2,p-1) & \xi^i{}_{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (p,1) & \ldots & (p,p-1) & \xi^i{}_{(p)} \end{vmatrix}$$

where

(1.5) $$D_0 = 1, \quad D_p = \begin{vmatrix} (1,1) & \ldots & (1,p) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (p,1) & \ldots & (p,p) \end{vmatrix}, \quad (p,q) = \xi_{(p)} \cdot \xi_{(q)}.$$

We shall now show that the set of n vectors defined by (1.4) satisfies the relation

(1.6) $$\eta_{(p)} \cdot \eta_{(q)} \equiv g_{ij}\, \eta^i{}_{(p)}\, \eta^j{}_{(q)} = \delta_{(pq)}$$

where $\delta_{(pq)}$ is the usual KRONECKER tensor.

It is easy to see that $\xi_{(q)} \cdot \eta_{(p)} = 0$ for $q < p$ and since $\eta_{(q)}$ is expressible as a linear combination of $\xi_{(1)}, \ldots, \xi_{(q)}$ it follows that

$$\eta_{(q)} \cdot \eta_{(p)} = 0 \quad \text{for} \quad q < p$$

so that the set of vectors (1.4) satisfies (1.6) for $q \neq p$. Also

$$\eta_{(p)} \cdot \eta_{(p)} = \frac{1}{D_{p-1}D_p} \begin{vmatrix} (1,1) & \ldots & (1,p-1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (p-1,1) & \ldots & (p-1,p-1) \end{vmatrix} \begin{vmatrix} (1,1) & \ldots & (1,p) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (p,1) & \ldots & (p,p) \end{vmatrix} = 1.$$

in view of (1.5).

Hence equation (1.6) is satisfied and any vector of the space must be expressible as a linear combination of the n vectors (1.4). Therefore we put

(1.7) $$\frac{D\,\eta^i{}_{(p)}}{Ds} = \sum_{q=1}^{n} C_{(pq)}\,\eta^i{}_{(q)}$$

where $C_{(pq)}$ is to be determined. From (1.4) and (1.7), we obtain

(1.8) $$C_{(pq)} = 0 \qquad \text{for} \qquad q > p+1.$$

Also differentiating (1.6) intrinsically, we obtain

$$g_{\underline{ij}}\left(\frac{D\,\eta^i{}_{(p)}}{Ds}\,\eta^j{}_{(q)} + \eta^i{}_{(p)}\,\frac{D\,\eta^j{}_{(q)}}{Ds}\right) = 0.$$

Now making use of (1.7), we obtain

(1.9) $$C_{(pq)} + C_{(qp)} = 0.$$

From (1.8) and (1.9), we have

(1.8a) $$C_{(pq)} = 0 \qquad \text{for} \qquad q < p-1.$$

Putting $p = q$ in (1.9),

(1.10) $$C_{(pp)} = 0.$$

In view of (1.8), (1.8a), (1.9) and (1.10), (1.7) simplifies to

(1.11) $$\frac{D\,\eta^i{}_{(p)}}{Ds} = -\,C_{(p-1,p)}\,\eta^i{}_{(p-1)} + C_{(p,p+1)}\,\eta^i{}_{(p+1)}\,.$$

We define the p th curvature of the curve C by

(1.12) $$K_p = C_{(p,p+1)} \qquad \text{with} \qquad K_0 = K_n = 0.$$

From (1.11) and (1.12), we obtain

(1.13) $$\frac{D\,\eta^i{}_{(p)}}{Ds} = -\,K_{p-1}\,\eta^i{}_{(p-1)} + K_p\,\eta^i{}_{(p+1)}.$$

We call (1.13) FRENET's *formulae*. We shall now show that

$$K_p = \frac{\sqrt{D_{p-1}\,D_{p+1}}}{D_p}\,.$$

From (1.6) and (1.13), we have

$$K_p = g_{\underline{ij}} \frac{D\eta^i{}_{(p)}}{Ds} \eta^j{}_{(p+1)}. \tag{1.14}$$

Now, differentiating (1.4) intrinsically, we obtain

$$\frac{D\eta^i{}_{(p)}}{Ds} = \sqrt{D_{p-1} D_p} \frac{D}{Ds}\left(\frac{1}{\sqrt{D_{p-1}D_p}}\right)\eta^i{}_{(p)}$$

$$+ \frac{1}{\sqrt{D_{p-1}D_p}}\left[\begin{vmatrix} (2,1) & \dots & (1,p-1) & \xi^i{}_{(1)} \\ (3,1) & \dots & (2.p-1) & \xi^i{}_{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (p+1,1) & \dots & (p,p-1) & \xi^i{}_{(p)} \end{vmatrix} + \cdots\right.$$

$$+\begin{vmatrix} (1,1) & \dots & (2,p-1) & \xi^i{}_{(1)} \\ (2,1) & \dots & (3,p-1) & \xi^i{}_{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (p,1) & \dots & (p+1,p-1) & \xi^i{}_{(p)} \end{vmatrix} + \begin{vmatrix} (1,1) & \dots & (1,p-2) & (1,p) & \xi^i{}_{(1)} \\ (2,1) & \dots & (2,p-2) & (2,p) & \xi^i{}_{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (p,1) & \dots & (p,p-2) & (p,p) & \xi^i{}_{(p)} \end{vmatrix}$$

$$\left. + \begin{vmatrix} (1,1) & \dots & (1,p-1) & \xi^i{}_{(2)} \\ (2,1) & \dots & (2,p-1) & \xi^i{}_{(3)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (p,1) & \dots & (p,p-1) & \xi^i{}_{(p+1)} \end{vmatrix} \right].$$

Multiplying both sides by $g_{\underline{ij}}\eta^i{}_{(p+1)}$ and making use of the fact that

$$\xi_{(q)} \cdot \eta_{(p+1)} = 0 \qquad \text{for} \qquad q = 1, \dots, p,$$

we obtain

$$K_p = \begin{vmatrix} (1,1) & \dots & (1,p-1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (p-1,1) & \dots & (p-1,p-1) \end{vmatrix} g_{\underline{ij}}\, \xi^i{}_{(p+1)}\, \eta^j{}_{(p+1)}$$

$$= \frac{1}{\sqrt{D_{p-1}D_p}} D_{p-1} \frac{1}{\sqrt{D_p D_{p+1}}} \begin{vmatrix} (1,1) & \dots & (1,p) & (1,p+1) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (p+1,1) & \dots & (p+1,p) & (p+1,p+1) \end{vmatrix} \tag{1.15}$$

$$= \frac{D_{p-1}}{D_p\sqrt{D_{p-1}D_{p+1}}} \cdot D_{p+1} = \frac{\sqrt{D_{p-1}D_{p+1}}}{D_p}.$$

SAXENA and BEHARI [2] obtained FRENET's formulae in the form

$$\frac{D\eta^i_{(p)}}{Ds} = -\overline{K}_{p-1}\,\eta^i_{(p-1)} + \overline{K}_p\,\eta^i_{(p+1)} + \underset{\smile}{\Lambda^i_{hj}}\,\eta^h_{(p)}\,t^j \tag{1.16}$$

where

$$\overline{K}_p = g_{\underline{ij}}\,\frac{\Delta\,\eta^i_{(p)}}{\Lambda s}\,\eta^j_{(p+1)}, \tag{1.17}$$

$\Lambda/\Lambda s$ denoting the intrinsic derivative with respect to the CHRISTOFFEL symbols formed by means of $g_{\underline{ij}}$.

We shall now establish the equivalence between (1.13) and (1.16). From (1.12) and (1.17), we obtain

$$K_p = \overline{K}_p + \underset{\smile}{\Lambda_{hkj}}\,\eta^h_{(p)}\,t^k\,\eta^j_{(p+1)}, \tag{1.18}$$

$\underset{\smile}{\Lambda_{hkj}}$ denoting the skew-symmetric part of Λ_{hkj}. In view of (1.18) the condition for the equivalence of (1.13) and (1.16) is given by

$$\underset{\smile}{\Lambda_{hkj}}\,\eta^h_{(p)}\,t^k\,[g^{\underline{ij}} - \eta^i_{(p+1)}\,\eta^j_{(p+1)} - \eta^i_{(p-1)}\,\eta^j_{(p-1)}] = 0, \tag{1.19}$$

$g^{\underline{ij}}$ being the conjugate tensor of $g_{\underline{ij}}$, we obtain [1]

$$g^{\underline{ij}} = \sum_q \eta^i_{(q)}\,\eta^j_{(q)}, \tag{1.20}$$

(1.16) may also be written as

$$\frac{\Delta\,\eta^i_{(p)}}{\Lambda s} = -\overline{K}_{p-1}\,\eta^i_{(p-1)} + \overline{K}_p\,\eta^i_{(p+1)}$$

so that

$$g_{\underline{ij}}\,\frac{\Lambda\,\eta^i_{(p)}}{\Lambda s}\,\eta^i_{(q)} = 0 \quad \text{except for} \quad q = p-1 \quad \text{and} \quad q = p+1\,. \tag{1.21}$$

Also from (1.13), we have

$$g_{\underline{ij}}\,\frac{D\,\eta^i_{(p)}}{Ds}\,\eta^j_{(q)} = 0 \quad \text{for} \quad q \neq p-1, \quad q \neq p+1. \tag{1.21a}$$

Subtracting (1.21) from (1.21a), we obtain

$$\underset{\smile}{\Lambda_{hkj}}\,\eta^h_{(p)}\,\eta^j_{(q)}\,t^k = 0 \quad \text{for} \quad q \neq p-1, \quad q \neq p+1. \tag{1.22}$$

From (1.20) and (1.22), we obtain

$$\underset{\smile}{\Lambda_{hkj}}\,\eta^h_{(p)}\,t^k\,g^{\underline{ij}} = \Lambda_{hkj}\,\eta^h_{(p)}\,t^k\,[\eta^i_{(p-1)}\,\eta^j_{(p-1)} + \eta^i_{(p+1)}\,\eta^j_{(p+1)}]$$

which shows that equation (1.19) is satisfied. Hence the two forms of FRENET's formulae, *viz.* (1.13) and (1.16) are equivalent.

2. Curvatures

Consider a curve $C: x^i = x^i(s)$ referred to its arc length as a parameter. We shall denote by $\eta^i_{(p)}$ and $\eta^{*i}_{(p)}$ respectively the components of the n orthogonalised vectors at two neighbouring points P and P^* of C. Let $\overline{\eta}^i_{(p)}$ be the components of the vector at P^* obtained from the corresponding vector $\eta^i_{(p)}$ at P by parallel displacement. $g_{\underline{ij}}$, $g^*_{\underline{ij}}$ will be used to denote the components of the symmetric part of the metric tensor at P and P^* respectively. The angle $\delta\vartheta_p$ between $\eta^{*i}_{(p)}$ and $\overline{\eta}^i_{(p)}$ is given by

$$\cos \delta\vartheta_p = g^*_{\underline{ij}}\, \eta^{*i}_{(p)}\, \overline{\eta}^j_{(q)} \tag{2.1}$$

By TAYLOR's expansion, we have

$$g^*_{\underline{ij}} = g_{\underline{ij}} + \left(\frac{d g^*_{\underline{ij}}}{ds}\right)_0 (\delta s) + \frac{1}{2}\left(\frac{d^2 g^*_{\underline{ij}}}{ds^2}\right)_0 (\delta s)^2 + \cdots \tag{2.2}$$

$$\eta^{*i}_{(p)} = \eta^i_{(p)} + \left(\frac{d\eta^{*i}_{(p)}}{ds}\right)_0 (\delta s) + \frac{1}{2}\left(\frac{d^2\eta^{*i}_{(p)}}{ds^2}\right)_0 (\delta s)^2 + \cdots \tag{2.3}$$

$$\overline{\eta}^i_{(p)} = \eta^i_{(p)} + \left(\frac{d\overline{\eta}^i_{(p)}}{ds}\right)_0 (\delta s) + \frac{1}{2}\left(\frac{d^2\overline{\eta}^i_{(p)}}{ds^2}\right)_0 (\delta s)^2 + \cdots . \tag{2.4}$$

From FRENET's formulae (1.13), we have

$$\frac{d\eta^{*i}_{(p)}}{ds} = -K^*_{p-1}\, \eta^{*i}_{(p-1)} + K^*_p\, \eta^{*i}_{(p+1)} - \Delta^{*i}_h\, \eta^{*h}_{(p)} \tag{2.5}$$

where $P^{\cdot i}_h = P^{\cdot i}_{hk}\, t^k$ and $*$ is used to indicate the value of the quantity at P^*. The unstarred quantities will represent the values of the same quantity at P.

Since $\overline{\eta}^i_{(p)}$ is obtained from $\eta^i_{(p)}$ by parallel transport, we have

$$\frac{d\overline{\eta}^i_{(p)}}{ds} = -\Delta^{\cdot i}_h\, \overline{\eta}^h_{(p)} . \tag{2.6}$$

Also $(D g_{\underline{ij}} / Ds) = 0$ [3] gives us

$$\frac{d g^*_{\underline{ij}}}{ds} = g^*_{\underline{ih}}\, \Delta^{*h}_j + g^*_{hj}\, P^{*h}_i . \tag{2.7}$$

Differentiating (2.5), (2.6) and (2.7) with respect to s and making use of the same equations, we obtain

$$\frac{d^2\eta^{*i}{}_{(p)}}{ds^2} = -\left(\frac{dK^*{}_{p-1}}{ds}\delta^i{}_h - \Delta^{*i}{}_h K^*{}_{p-1}\right)\eta^{*h}{}_{(p-1)} + \left(\frac{dK^*{}_p}{ds}\delta^i{}_h - \Delta^{*i}{}_h K^*{}_p\right)\eta^{*h}{}_{(p+1)}$$

(2.8)
$$-\frac{d\Delta^{*i}{}_h}{ds}\eta^{*h}{}_{(p)} + K^*{}_{p-1}K^*{}_{p-2}\,\eta^{*i}{}_{(p-2)} + K^*{}_p K^*{}_{p+1}\,\eta^{*i}{}_{(p+2)}$$
$$-(K^{*2}{}_{p-1} + K^{*2}{}_p)\,\eta^{*i}{}_{(p)}$$

(2.9)
$$\frac{d^2\eta^i{}_{(p)}}{ds^2} = -\frac{d\Delta^{*i}{}_h}{ds}\bar{\eta}^h{}_{(p)} + \Delta^{*i}{}_h\Delta^{*h}{}_k\,\bar{\eta}^k{}_{(p)}$$

(2.10)
$$\frac{d^2 g^*{}_{\underline{ij}}}{ds^2} = g^*{}_{\underline{ih}}\frac{d\Delta^{*h}{}_j}{ds} + g^*{}_{\underline{hj}}\frac{d\Delta^{*h}{}_i}{ds} + \Delta^{*h}{}_j(\Delta^*{}_{ih} + \Delta^*{}_{hi}) + \Delta^{*h}{}_i(\Delta^*{}_{hj} + \Delta^*{}_{jh})$$

where

(2.11)
$$\Delta^*{}_{ij} = g^*{}_{\underline{hj}}\,\Delta^{*h}{}_i\,.$$

From (2.1), (2.2), (2.3) and (2.4) we obtain

$$\cos\delta\vartheta_p = 1 + \left[g_{\underline{ij}}\left(\frac{d\eta^{*i}{}_{(p)}}{ds} + \frac{d\bar{\eta}^i{}_{(p)}}{ds}\right)_0 \eta^j{}_{(p)} + \left(\frac{dg^*{}_{\underline{ij}}}{ds}\right)_0 \eta^i{}_{(p)}\,\eta^j{}_{(p)}\right](\delta s)$$

(2.12)
$$+\left[g_{\underline{ij}}\left(\frac{d\eta^{*i}{}_{(p)}}{ds}\right)_0\left(\frac{d\bar{\eta}^j{}_{(p)}}{ds}\right)_0 + \frac{1}{2}g_{\underline{ij}}\left(\frac{d^2\eta^{*i}{}_{(p)}}{ds^2} + \frac{d^2\bar{\eta}^i{}_{(p)}}{ds^2}\right)_0\eta^j{}_{(p)}\right.$$
$$\left.+\left(\frac{dg^*{}_{\underline{ij}}}{ds}\right)_0\left(\frac{d\eta^{*i}{}_{(p)}}{ds} + \frac{d\bar{\eta}^i{}_{(p)}}{ds}\right)_0\eta^j{}_{(p)} + \frac{1}{2}\left(\frac{d^2 g^*{}_{\underline{ij}}}{ds^2}\right)_0\eta^i{}_{(p)}\,\eta^j{}_{(p)}\right](\delta s)^2 + \cdots$$

Substituting the values of the various quantities from equations (2.5) to (2.10) in (2.12) and making use of the orthogonality relations (1.6), we obtain

$$\cos\delta\vartheta_p = 1 - \frac{1}{2}(K^2{}_{p-1} + K^2{}_p)(\delta s)^2 + \cdots$$

which may be written as

$$\frac{1-\cos^2\delta\vartheta_p}{(\delta\vartheta_p)^2}\left(\frac{\delta\vartheta_p}{\delta s}\right)^2 = (K^2{}_{p-1} + K^2{}_p) + \text{terms containing powers of } (\delta s).$$

Taking limits as $\delta s \to 0$, we obtain

(2.13)
$$\left(\frac{d\vartheta_p}{ds}\right)^2 = K^2{}_{p-1} + K^2{}_p\,.$$

If ϱ_p denotes the p th radius of curvature, then

$$\varrho_p = \frac{1}{K_p}$$

so that (2.13) assumes the form

$$\left(\frac{d\vartheta_p}{ds}\right)^2 = \frac{1}{\varrho^2_{p-1}} + \frac{1}{\varrho^2_p}. \tag{2.14}$$

The same result was established by LEVY [4] for a RIEMANN space. Our definition of ϱ_p coincides with that of LEVY when $\underset{\smile}{g_{ij}} = 0$ so that the corresponding result for a RIEMANN space follows from (2.14) as a particular case.

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DEPARTMENT OF MATHEMATICS
KIRORI MAL COLLEGE
UNIVERSITY OF DELHI
DELHI — INDIA

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ÖZET

n-boyutlu RIEMANN uzayları için WEATHERBURN [1] tarafından elde edilen FRENET formüllerinin teşmil edilmiş şekli umumileştirilmiş RIEMANN uzaylarına SAXENA ve RAM BEHARI [2] tarafından teşmil edilmiş bulunmakladır. Bu araştırmada ise umumileştirilmiş bir RIEMANN uzayı için FRENET formüllerinin diğer bir şekli elde edilmiş, ve bunun yukarda zikredilen şekille intibak ettiğini de gösterilmiştir. Müteakip iki binormal arasındaki açının türevinin, ait oldukları eğrinin yay uzunluğu ve eğrilikleri cinsinden RIEMANN uzaylarında muteber olan ve LEVY [4] tarafından verilen bir ifadeleri de umumileştiriimiş RIEMANN uzaylarına teşmil edilmiş bulunmaktadır.