STEADY FLOW OF A CONDUCTING NON-NEWTONIAN FLUID IN AN ANNULUS UNDER A RADIAL MAGNETIC FIELD AND WITH SUCTION AND INJECTION

J. N. KAPUR AND R. K. RATHY

In this paper the flow of a conducting non-Newtonian fluid in an annulus under a radial magnetic field and constant pressure gradient is considered. The solution is obtained in powers of the cross-viscosity REYNOLDS number which is assumed to be small. The discussion includes cases when the walls are moving or stationary. The flow between parallel plates has been obtained as a limiting case.

1. Introduction :

The flow of a conducting fluid in an annulus has been discussed by GLOBE [1]. Later CHECKMERAV I. B. [2] and R. K. JAIN [3] have independently extended the problem to include the case of constant suction and injection. In this paper the flow of a non-Newtonian conducting fluid through an annulus under a radial magnetic field with constant suction and injection and with a constant pressure gradient, is considered. The polar coordinates $(\bar{r}, \vartheta, z)$ are used. As the problem concerns only the steady and axially symmetric flow, the variables are independent of time t and of ϑ. It is also assumed that the flow does not depend on z and the main flow is in the z-direction. The external radial magnetic field is assumed to be of the form $H_0 R/\bar{r}$. Again the displacement currents and free charge density are also assumed to be zero.

Let ϱ, σ, μ_e, μ, μ_c denote the density, electrical conductivity, magnetic permeability, viscosity and crossviscosity of the fluid respectively and $\mathbf{H}(H_0 R/\bar{r}, 0, H_z)$, $\mathbf{V}(v_r, 0, v_z)$, $\mathbf{E}(0, E_\vartheta, 0)$ and $\mathbf{J}(0, J_\vartheta, 0)$ are the magnetic, velocity, current and electric field vectors.

2. Basic equations and their solutions :

With the above assumptions the basic equations reduce to

$$(1) \qquad v_r \frac{\partial v_r}{\partial \bar{r}} = -\frac{1}{\varrho}\frac{\partial p}{\partial \bar{r}} + \frac{\mu_c}{\varrho}\left[2\frac{\partial}{\partial \bar{r}}\left(\frac{v_r}{\bar{r}}\right)^2 + \frac{\partial}{\partial \bar{r}}\left(\frac{v_z}{\bar{r}}\right)^2\right] - \frac{\mu_e}{4\pi\varrho} H_z \frac{\partial H_z}{\partial \bar{r}},$$

$$(2)\qquad v_r \frac{\partial v_z}{\partial \bar{r}} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{\mu}{\varrho}\left(\frac{\partial^2 v_z}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial v_z}{\partial \bar{r}}\right) - \frac{\mu_c}{\varrho}\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(v_r \frac{\partial v_z}{\partial \bar{r}}\right)\right] + \frac{\mu_e}{4\pi\varrho}\frac{H_0 R}{\bar{r}}\frac{\partial H_z}{\partial \bar{r}},$$

$$(3)\qquad -\frac{\partial H_z}{\partial \bar{r}} = 4\pi\sigma\left[E_\vartheta + \mu_e \frac{H_0 R}{\bar{r}} v_z - \mu_e v_r H_z\right].$$

We now reduce the above equations with the help of the relations

$$v_r = \frac{v_0 R}{\bar{r}}, \quad v_z = v_0 u, \quad \bar{r} = R \cdot r, \quad H_z = H_0 h,$$

$$(4)\qquad \nu = \frac{\mu}{\varrho R v_0}, \quad \nu_m = \frac{1}{4\pi\sigma\mu_e R v_0}, \quad \nu_c = \frac{\mu_c}{\varrho R^2},$$

$$M = \mu_e H_0 R \sqrt{\frac{\sigma}{\mu}}, \quad E = \frac{E_\vartheta}{\mu_e H_0 v_0} \quad \text{and} \quad P = -\frac{R}{\varrho v_0^2}\frac{\partial p}{\partial z},$$

to non-dimensional form and integrate equation (1). We obtain

$$(5)\qquad p = \varrho v_0^2\left[\nu_c\left(\frac{2}{r^4} + \frac{u^2}{r^2}\right) - \frac{1}{2r^2} - M^2 \nu \nu_m \frac{h^2}{2}\right] + \text{Const.}$$

$$(6)\qquad \frac{1}{r}\frac{du}{dr} = P + \nu\left(\frac{du^2}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) - \frac{\nu_c}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{du}{dr}\right) + \frac{M^2 \nu \nu_m}{r}\frac{dh}{dr},$$

$$(7)\qquad u = -r\nu_m \frac{dh}{dr} - E r.$$

Multiplying equation (6) by r and on integration we find,

$$(8)\qquad u = \frac{1}{2} P r^2 + \nu r \frac{du}{dr} - \frac{\nu_c}{r}\frac{du}{dr} + M^2 \nu \nu_m h + c'.$$

2a. Solution

We now assume ν_c to be small and therefore we can expand u, h, E and c' in the terms of ν_c as

$$(9)\qquad u = \sum_{n=0}^{\infty} \nu_c^n u_n, \quad h = \sum_{n=0}^{\infty} \nu_c^n h_n, \quad E = \sum_{n=0}^{\infty} \nu_c^n E_n, \quad c' = \sum_{n=0}^{\infty} \nu_c^n c'_n.$$

Boundary conditions : We take aR and bR as the radii of the inner and outer cylinders respectively. When the boundaries are fixed, we have the boundary conditions as

(10) at $r = a, \quad u = 0; \quad h = 0 \quad i.e. \quad \begin{matrix} u_0 = u_1 = \cdots = 0 \\ h_0 = h_1 = \cdots = 0, \end{matrix}$

(11) at $r = b, \quad a = 0, \quad h = 0 \quad i.e. \quad \begin{matrix} u_0 = u_1 = \cdots = 0 \\ h_0 = h_1 = \cdots = 0. \end{matrix}$

Zero order solution: Taking only the zeroth order terms we get from equations (7) and (8)

$$u_0 = \frac{1}{2} P r^2 + \nu r \frac{du_0}{dr} + M^2 \nu \nu_m h_0 + c_0', \tag{12}$$

and

$$u_0 = h_0 - r \nu_m \frac{dh_0}{dr} - E_0 r. \tag{13}$$

Eliminating u_0 between (12) and (13) and solving we obtain

$$h_0 = A_0 r^\alpha + B_0 r^\beta + \frac{P r^2}{2\nu \nu_m (\alpha-2)(\beta-2)} + \frac{E_0 (1-\nu) r}{\nu \nu_m (\alpha-1)(\beta-1)} + C_0. \tag{14}$$

Substituting the value of h_0 in equation (13) we get

$$u_0 = A_0 (1 - \alpha \nu_m) r^\alpha + B_0 (1 - \beta \nu_m) r^\beta + \frac{P(1-2\nu_m) r^2}{2\nu \nu_m (\alpha-2)(\beta-2)} + \frac{M^2 E_0 r}{(\alpha-1)(\beta-1)} + C_0 \tag{15}$$

where A_0 and B_0 are constants and

$$\alpha = \frac{1}{2}\left(\frac{1}{\nu} + \frac{1}{\nu_m}\right) + \frac{1}{2}\sqrt{4M^2 + \left(\frac{1}{\nu} - \frac{1}{\nu_m}\right)^2} \tag{16}$$

$$\beta = \frac{1}{2}\left(\frac{1}{\nu} + \frac{1}{\nu_m}\right) - \frac{1}{2}\sqrt{4M^2 + \left(\frac{1}{\nu} - \frac{1}{\nu_m}\right)^2}. \tag{17}$$

Using these boundary conditions we obtain the values of the constants as

$$A_0 = \frac{P}{2\nu\nu_m(\alpha-2)(\beta-2)\Delta} \Big[\{(1 - \nu - M^2 \nu \nu_m) a (b^2 - a^2) - 2\nu_m (1-\nu) a^2 (b-a)\} \times \{M^2 \nu \nu_m - (1-\nu)(1 - \beta \nu_m)\} (b^\beta - a^\beta) - \{(1 - \nu - M^2 \nu \nu_m) a (b^\beta - a^\beta) - \beta \nu_m (1-\nu) a^\beta (b-a)\} \{M^2 \nu \nu_m - (1 - 2\nu_m)(1-\nu)\} (b^2 - a^2)\Big], \tag{18}$$

$$A_0 = \frac{-P}{2\nu\nu_m(\alpha-2)(\beta-2)\Delta} \Big[\{(1 - \nu - M^2 \nu \nu_m) a (b^2 - a^2) - 2\nu_m (1-\nu) a^2 (b-a)\} \times \{M^2 \nu \nu_m - (1-\nu)(1 - \alpha \nu_m)\} (b^\alpha - a^\alpha) - \{M^2 \nu \nu_m - (1-\nu)(1 - 2\nu_m)\} \times \{a (1 - \nu - M^2 \nu \nu_m)(b^\alpha - a^\alpha) - \alpha \nu_m (1-\nu) a^\alpha (b-a)\} (b^2 - a^2)\Big], \tag{19}$$

(20) $$E_0 = -\frac{(1-\alpha)(1-\beta)\nu\nu_m}{(1-\nu)(b-a)}\left[A_0(b^\alpha - a^\alpha) + B_0(b^\beta - a^\beta) + \frac{P(b^2-a^2)}{2(2-\alpha)(2-\beta)\nu\nu_m}\right],$$

(21) $$C_0 = -\frac{1}{b-a}\left[A_0(a^\alpha b - a b^\alpha) + B_0(a^\beta b - a b^\beta) - \frac{P ab(b-a)}{2(2-\alpha)(2-\beta)\nu\nu_m}\right],$$

where

(22) $$\begin{aligned} \Delta = [\,&\{(1-\nu-M^2\nu\nu_m)\,a\,(b^\beta - a^\beta) - \beta\nu_m(1-\nu)\,a^\beta(b-a)\} \\ &\times\{M^2\nu\nu_m - (1-\nu)(1-\alpha\nu_m)\}(b^\alpha - a^\alpha) - \{(1-\nu-M^2\nu\nu_m)\,a\,(b^\alpha - a^\alpha) \\ &- \alpha\nu_m(1-\nu)\,a^\alpha(b-a)\}\{M^2\nu\nu_m - (1-\nu)(1-\beta\nu_m)\}(b^\beta - a^\beta)\,]. \end{aligned}$$

First order solution: We take the first order terms also into aeconnt in equations (7) and (8) and eliminating, we get

(23) $$r^2\frac{d^2h_1}{dr^2} + \left(1 - \frac{1}{\nu} - \frac{1}{\nu_m}\right) r\frac{dh_1}{dr} + \left(\frac{1}{\nu\nu_m} - M^2\right) h_1 = \frac{c_1'}{\nu\nu_m} - \frac{1}{r\nu\nu_m}\frac{du_0}{dr} + \frac{E_1 r(1-\nu)}{\nu\nu_m}.$$

Solving it we get

(24) $$h_1 = A_1 r^\alpha + B_1 r^\beta + C_1 + \frac{A_0\alpha(1-\alpha\nu_m)}{2(\alpha-\beta-2)\nu\nu_m} r^{\alpha-2} + \frac{B_0\beta(1-\beta\nu_m)}{2(\beta-\alpha-2)\nu\nu_m} r^{\beta-2} - \frac{M^2E_0}{\nu\nu_m(\alpha^2-1)(\beta^2-1)}\frac{1}{r} + \frac{E_1(1-\nu)r}{\nu\nu_m(\alpha-1)(\beta-1)}$$

where A_1, B_1 and C_1 are constants.

Substituting the value of h_1 in equation (7) and taking first order terms we get

(25) $$u_1 = A_1(1-\alpha\nu_m)r^\alpha + B_1(1-\beta\nu_m)r^\beta + C_1 + \frac{A_0\alpha(1-\alpha\nu_m)(1-\overline{\alpha-2}\nu_m)}{2(\alpha-\beta-2)\nu\nu_m} r^{\alpha-2} + \frac{B_0\beta(1-\beta\nu_m)(1-\overline{\beta-2}\nu_m)}{2(\beta-\alpha-2)\nu\nu_m} r^{\beta-2} - \frac{M^2E_0(1+\nu_m)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)}\frac{1}{r} + \frac{M^2E_1 r}{(\alpha-1)(\beta-1)}.$$

Using the boundary conditions, we get the values of the constants as

$$(26)\quad A_\iota = \frac{1}{\Delta}\Bigg[\Big[\frac{A_0(1-\alpha\nu_m)\alpha}{2(\alpha-\beta-2)\nu\nu_m}\{(1-\nu-M^2\nu\nu_m)a(b^{\alpha-2}-a^{\alpha-2})-(\alpha-2)\nu_m(1-\nu)$$

$$\times a^{\alpha-2}(b-a)\} + \frac{B_0(1-\beta\nu_m)\beta}{2(\beta-\alpha-2)\nu\nu_m}\{(1-\nu-M^2\nu\nu_m)a(b^{\beta-2}-a^{\beta-2})-(\beta-2)\nu_m(1-\nu)a^{\beta-2}(b-a)\}$$

$$+\frac{M^2E_0(b-a)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)}\left\{\frac{1}{b}(1-\nu-M^2\nu\nu_m)-\frac{1}{a}(1-\nu)\nu_m\right\}\Big]\Big[M^2\nu\nu_m-(1-\nu)(1-\beta\nu_m)\Big](b^\beta-a^\beta)$$

$$-\Big[\frac{\alpha(1-\alpha\nu_m)A_0}{2(\alpha-\beta-2)\nu\nu_m}\left\{M^2\nu\nu_m-(1-\nu)(1-\overline{\alpha-2}\nu_m)\right\}(b^{\alpha-2}-a^{\alpha-2})$$

$$+\frac{\beta(1-\beta\nu_m)B_0}{2(\beta-\alpha-2)\nu\nu_m}\left\{M^2\nu\nu_m-(1-\nu)(1-\overline{\beta-2}\nu_m)\right\}(b^{\beta-2}-a^{\beta-2})$$

$$+\frac{M^2E_0(b-a)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)ab}\left\{M^2\nu\nu_m-(1+\nu_m)(1-\nu)\right\}\Big]$$

$$\Big[(1-\nu-M^2\nu\nu_m)a(b^\beta-a^\beta)-\beta\nu_m(1-\nu)a^\beta(b-a)\Big]\Bigg],$$

$$(27)\quad B_\iota = \frac{-1}{\Delta}\Bigg[\Big[\frac{A_0\alpha(1-\alpha\nu_m)}{2(\alpha-\beta-2)\nu\nu_m}\left\{(1-\nu-M^2\nu\nu_m)a(b^{\alpha-2}-a^{\alpha-2})-(\alpha-2)(1-\nu)\nu_m a^{\alpha-2}(b-a)\right\}$$

$$+\frac{B_0\beta(1-\beta\nu_m)}{2(\beta-\alpha-2)\nu\nu_m}\left\{(1-\nu-M^2\nu\nu_m)a(b^{\beta-2}-a^{\beta-2})-(\beta-2)(1-\nu)\nu_m a^{\beta-2}(b-a)\right\}$$

$$+\frac{M^2E_0(b-a)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)}\left\{\frac{1}{b}(1-\nu-M^2\nu\nu_m)-\frac{\nu_m}{a}(1-\nu)\right\}\Big]\Big[M^2\nu\nu_m-(1-\nu)(1-\alpha\nu_m)\Big]$$

$$\times(b^\alpha-a^\alpha)-\Big[\frac{\alpha(1-\alpha\nu_m)A_0}{2(\alpha-\beta-2)\nu\nu_m}\left\{M^2\nu\nu_m-(1-\nu)(1-\overline{\alpha-2}\nu_m)\right\}(b^{\alpha-2}-a^{\alpha-2})$$

$$+\frac{\beta(1-\beta\nu_m)B_0}{2(\beta-\alpha-2)\nu\nu_m}\left\{M^2\nu\nu_m-(1-\nu)(1-\overline{\beta-2}\nu_m)\right\}(b^{\beta-2}-a^{\beta-2})+\frac{M^2E_0(b-a)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)ab}$$

$$\times\left\{M^2\nu\nu_m-(1+\nu_m)(1-\nu)\right\}\Big]\Big[(1-\nu-M^2\nu\nu_m)a(b^\alpha-a^\alpha)-\alpha\nu_m(1-\nu)a^\alpha(b-a)\Big]\Bigg],$$

$$(28)\quad E_1 = \frac{-\nu\nu_m(\alpha-1)(\beta-1)}{(1-\nu)(b-a)}\Big[A_1(b^\alpha-a^\alpha)+B_1(b^\beta-a^\beta)+\frac{\alpha(1-\alpha\nu_m)A_0}{2(\alpha-\beta-2)\nu\nu_m}$$

$$\times(b^{\alpha-2}-a^{\alpha-2})+\frac{\beta(1-\beta\nu_m)B_0}{2(\beta-\alpha-2)\nu\nu_m}(b^{\beta-2}-a^{\beta-2})+\frac{M^2E_0(b-a)}{\nu\nu_m(\alpha^2-1)(\beta^2-1)ab}\Big],$$

$$(29)\quad C_1 = -\Big[A_1a^\alpha+B_1a^\beta+\frac{\alpha(1-\alpha\nu_m)A_0}{2(\alpha-\beta-2)\nu\nu_m}a^{\alpha-2}+\frac{\beta(1-\beta\nu_m)B_0}{2(\beta-\alpha-2)\nu\nu_m}a^{\beta-2}$$

$$-\frac{M^2E_0}{\nu\nu_m(\alpha^2-1)(1-\beta^2)}\frac{1}{a}+\frac{E_1(1-\nu)}{\nu\nu_m(\alpha-1)(\beta-1)}a\Big].$$

*N*th order solution: Taking n th order terms also into account in the equations (7) and (8) and eliminating u_n we get

$$r^2 \frac{d^2 h_n}{dr^2} + \left(1 - \frac{1}{\nu} - \frac{1}{\nu_m}\right) r \frac{dh_n}{dr} + \left(\frac{1}{\nu \nu_m} - M^2\right) h_n = \frac{C_n'}{\nu \nu_m} - \frac{1}{\nu \nu_m r} \frac{dv_{n-1}}{dr} + \frac{E_n (1-\nu) \nu}{\nu \nu_m}. \tag{30}$$

Thus knowing the solution upto $(n-1)$ th order we can find out n th order solution. This shows that we can find out the solution upto any order we like.

2b. When the walls are in motion:

Let the inner and the outer cylinders be moving with velocities $v_0 U_1$ and $v_0 U_2$ respectively. In this case the boundary conditions will be

$$\text{at} \quad r = a, \quad u = U_1, \quad h = 0, \quad i.\ e. \begin{cases} u_0 = U_1, \ u_1 = u_2 = \cdots = 0 \\ h_0 = h_1 = h_2 = \cdots = 0, \end{cases} \tag{31}$$

$$\text{at} \quad r = b, \quad u = U_2, \quad h = 0, \quad i.\ e. \begin{cases} u_0 = U_2, \ u_1 = u_2 = \cdots = 0 \\ h_0 = h_1 = h_2 = \cdots = 0. \end{cases} \tag{32}$$

Substituting these values we can find out the values of $A_0, B_0, C_0, E_0, A_1, B_1, E_1$ and C_1 and subsiquently we can find out the solution.

2c. Limiting case of flow throught parallel plates:

If we write

$$b = \frac{L+R}{R}$$

and

$$a = 1 - \frac{L}{R}, \quad \frac{L}{R} = \varepsilon$$

and let $\varepsilon \to 0$ we get the limiting case of plates. To obtain this we shall have to modify all the perameters and then to take the limit. The modification are:

$$\alpha = \frac{\alpha'}{\varepsilon}, \quad \beta = \beta'/\varepsilon, \quad \nu = \varepsilon \nu', \quad M = \frac{M'}{\varepsilon},$$

$$\nu_m = \varepsilon \nu_m', \quad \nu_c = \varepsilon^2 \nu_c', \quad r = 1 + \frac{L\eta}{R} \quad \text{and} \quad P = \frac{P'}{\varepsilon}. \tag{33}$$

Now substituting these values and taking the limits we obtain

(34) $$u = \frac{P'}{(M^2\nu\, \nu_m - 1)} \left[\frac{\beta'(1-\alpha'\nu_m')(\cosh\alpha' - e^{\alpha'\eta})}{(\alpha'-\beta')\sinh\alpha'} - \frac{\alpha'(1-\beta'\nu_m')(\cosh\beta' - e^{\beta'\eta})}{(\alpha'-\beta')\sinh\beta'} - \eta \right],$$

(35) $$h = \frac{P'}{(M^2\nu\, \nu_m - 1)} \left[\frac{\beta'(\cosh\alpha' - e^{\alpha'\eta})}{(\alpha'-\beta')\sinh\alpha'} - \frac{\alpha'(\cosh\beta' - e^{\beta'\eta})}{(\alpha'-\beta')\sinh\beta'} - \eta \right],$$

(36) $$E = \frac{P'\nu_m}{(M^2\nu\, \nu_m - 1)} \left[\frac{\alpha'\beta'\sinh(\alpha'-\beta')}{(\alpha'-\beta')\sinh\alpha'\sinh\beta'} - 1 \right].$$

2e. When the external field is zero

We get hydrodynamic flow as the limiting case if we take $M \to 0$ and $\nu_m \to \infty$. Taking the limit, we obtain

(43) $$u_0 = \frac{P}{2(1-2\nu)} \left[\frac{(b^2-a^2)(a^{1/\nu} - r^{1/\nu})}{(b^{1/\nu} - a^{1/\nu})} + r^2 - a^2 \right],$$

(44) $$u_1 = \frac{P(b^2-a^2)}{4\nu(2\nu-1)} \left[\frac{(b^{(1/\nu)-2} - a^{(1/\nu)-2})(r^{1/\nu} - a^{1/\nu})}{b^{1/\nu} - a^{1/\nu}} + a^{(1/\nu)-2} - r^{(1/\nu)-2} \right].$$

3. Conclusion

We find that the cross-viscosity affects the flow field and the induce magnetic field in the case of the flow through an annulus, while it has no effect on either, in the case of flow through parallel plates. In the later case the cross-viscosity only affects the pressure.

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
KANPUR — INDIA

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ÖZET

Yarıçap istikametinde tesirial gösteren bir manyetik alan ile sabit bir tazyik gradientinin tesiri altında iletken gayrı-newtoniyen bir akışkanın bir halkadaki akışı göz önüne alınmış. Çözüm, küçük olduğu kabul edilen, kesit viskozitesi REYNOLDS sayısının kuvvetleri cinsinden bir açılımla ifade edilmiştir. Neticenin irdelemesinde davarların sabit veya hareket etmeleri halleri incelenmiştir. Paralel levhalar arasındaki akış yukardaki neticelerin bir limit halı olarak elde edilmiştir.