FRENET'S FORMULAE AND CURVATURES IN A GENERALISED FINSLER SPACE

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Generalised FINSLER spaces of the second kind were studied in [3]. In the present note, FRENET's formulae have been obtained for an n-dimensional generalised FINSLER space of the second kind. The FRENET's formulae for a FINSLER space [4] and for a generalised RIEMANN space [2] follow as a particular case. A result regarding curvatures obtained by the author tor generalised Riemaanian spaces has been extended to generalised FINSLER spaces. A similar result of KAUL [5] for FINSLER spaces follows as a particular case.

1. **Introduction.** Let F_n be an n-dimensional generalised FINSLER manifold endowed with a local coordinate system. To each point P of F_n is associated a non-symmetric metric tensor $g_{ij}(x, \dot{x})$, $(x, \dot{x})$ being the element of support, so that the distance between two neighbouring points is given by

$$ds^2 = g_{ij}(x, \dot{x})\, dx^i\, dx^j . \tag{1.1}$$

If $F(x, \dot{x})$ be the distance function, *i. e.* if

$$ds = F(x, dx), \tag{1.1a}$$

where F satisfies the usual conditions [3], then we have

$$g_{(ij)}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i\, \partial \dot{x}^j}, \tag{1.2}$$

$g_{(ij)}$ denoting the symmetric part of $g_{ij}(x, \dot{x})$. The skew-symmetric part of g_{ij} will be denoted by $g_{[ij]}$.

(*) The Author wishes to thank Dr. RAM BEHARI and Dr. P. B. BHATTACHARYA for their guidance, encouragement and inspiration during this work, and Dr. P. K. KELKAR and Dr. J. N. KAPUR for providing him with research facilities.

The intrinsic derivative of a contravariant vector field X^i along a curve $C: x^i = x^i(s)$ is given by

$$\frac{Dx^i}{Ds} = \frac{dx^i}{ds} + \Gamma^i_{hk} X^h \frac{dx^k}{ds} + C^i_{hk} X^h \frac{dx^k}{ds}, \tag{1.3}$$

where

$$\Gamma^i_{hk} = \{^i_{hk}\} + g^{(im)} A_{hkr} \Gamma^r_{om} - A^i_{kr} \Gamma^r_{oh}, \tag{1.4}$$

A^i_{hk} being defined by

$$A^i_{hk} = g^{(ij)} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^h} - \frac{\partial g_{hk}}{\partial x^j} \right). \tag{1.4a}$$

It is proved in [3] that

$$\frac{D}{Ds} [g_{(ij)}(x, x^*)] = 0. \tag{1.5}$$

The infinitesimal parallelism is, as usual, defined by the vanishing of the intrinsic derivative in the direction of the curve.

2. FRENET's formulae. We define a set of n vectors $\xi^i_{p/}$ in the following manner:

$$\xi^i_{1/} = \frac{dx^i}{ds}, \qquad \xi^i_{p/} = \frac{D}{Ds} \xi^i_{p-1/} \qquad (p = 2, \ldots, n) \tag{2.1}$$

D/Ds denoting the intrinsic derivative in the direction of the curve C. Evidently $\xi^i_{1/}$ are the contravariant components of the unit tangent to C. We define another set of n vectors $\eta^i_{p/}$ as follows:

$$\eta^i_{1/} = \xi^i_{1/},$$

$$\eta^i_{p/} = \frac{1}{\sqrt{D_{p-1} D_p}} \begin{vmatrix} (1,1) & \ldots & (1, p-1) & \xi^i_{1/} \\ (2,1) & \ldots & (2, p-1) & \xi^i_{2/} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (p,1) & \ldots & (p, p-1) & \xi^2_{p/} \end{vmatrix} \quad (p = 2, \ldots, n) \tag{2.2}$$

where

$$D_0 = 1, \; D_p = \begin{vmatrix} (1,1) & \ldots & (1,p) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ (p,1) & \ldots & (p,p) \end{vmatrix} \quad \text{and} \quad (p, q) = \xi_{p/i} \, \xi^i_{q/}. \tag{2.3}$$

It may be verified by direct calculation that

$$\eta_{p/i}\, \eta^i_{q/} = \delta_{pq/} \tag{2.4}$$

where $\delta_{pq/}$ is 1 or 0 according as $p = q$ or $p \neq q$,

$\eta^i_{1/}, \eta^i_{2/}, \ldots, \eta^i_{n/}$ are called the unit tangent, unit principal normal, unit first binormal, ..., unit $n-2$th binormal respectively.

Any vector lying in F_n must be expressible as a linear combination of the n vectors (2.2). So we write

$$\frac{D\,\eta^i_{p/}}{Ds} = \sum_q C_{pq/}\, \eta^i_{q/}, \qquad p = 1, \ldots, n \tag{2.5}$$

where $C_{pq/}$ are to be determined. From (2.2) and (2.5) we obtain

$$C_{pq/} = 0 \quad \text{for} \quad q > p+1. \tag{2.6}$$

Also, taking the dot product of (2.5) with $\vec{\eta}_{q/}$, we obtain

$$g_{(ij)}(x, \dot{x}) \frac{D\eta^i_{p/}}{Ds} \eta^j_{q/} = C_{pq/}. \tag{2.7}$$

Differentiating (2.3) intrinsically and making use of (1.5) and (2.7), we obtain

$$C_{pq/} + C_{qp/} = 0 \tag{2.8}$$

For $q = p$, we obtain

$$C_{pp/} = 0. \tag{2.8a}$$

From (2.6) and (2.8), we obtain

$$C_{pq/} = 0 \quad \text{for} \quad q < p-1 \tag{2.8b}$$

Equations (2.6), (2.8a) and (2.8b) may be combined as

$$C_{pq/} = 0 \quad \text{for} \quad q \neq p-1 \quad \text{and} \quad q \neq p+1. \tag{2.9}$$

From (2.5) and (2.9), we obtain

$$\frac{D\,\eta^i_{p/}}{Ds} = C_{p,\,p-1/}\, \eta_{p-1/} + C_{p,\,p+1/}\, \eta^i_{p+1/} \tag{2.10}$$

with

$$C_{p0/} = C_{n,n+1/} = 0.$$

We now define the curvatures of the curve C by the following equations:

$$k_p = C_{p,\, p+1/} = - C_{p+1,\, p/} = g_{(ij)} \frac{D \eta^i_{p/}}{Ds} \eta^j_{p+1/} \tag{2.11}$$

so that (2.10) takes the form

$$\frac{D \eta^i_{p/}}{Ds} = - k_{p-1} \eta^i_{p-1/} + k_p \eta^i_{p+1} \tag{2.12}$$

with

$$k_0 = k_n = 0.$$

It follows by an easy direct calculation similar to the one performed in [2] that

$$k_p = \frac{\sqrt{D_{p-1} D_{p+1}}}{D_p}. \tag{2.13}$$

Equations (2.13) will be valid for $p = 0$ and n also if we define

$$D_{-1} = D_{n+1} = 0. \tag{2.13a}$$

Equations (2.12), represent FRENET's formulae. When $g_{(ij)} = 0$, D/Ds is the same as the ϑ-operator of TAYLOR [1] and equations (2.12) represent FRENET's formulae for a FINSLER space. On the other hand, if the space be a generalised RIEMANNIAN space, then $\Gamma^i_{hk} = \Delta^i_{hk}$ and D/Ds is the same as the intrinsic derivative for a generalised RIEMANNIAN space [6] and (2.12) represent FRENET's formulae as obtained in [2].

3. Curvatures. Let us consider a curve $C : x^i = x^i(s)$ referred to its arc length s. We shall denote by $\eta^i_{p/}$ and $\eta^{*i}_{p/}$ respectively the components of the nth orthogonalised vectors at consecutive points P and P^* of C. $\overline{\eta}^i_{p/}$ will be used to denote the vector at P^* obtained by the infinitesimal parallel displacement of the corresponding vector $\eta^i_{p/}$ at P. $g_{(ij)}$, $g^*_{(ij)}$ will denote the components of the symmetric part of the metric tensor at P and P^* respectively. If

$$\delta \vartheta_p \qquad (p = 1, \ldots, n)$$

denote the angle between $\eta^{*i}_{p/}$ and $\overline{\eta}^i_{p/}$, then we have [3]

$$\cos \delta \vartheta_p = g^*_{(ij)} (x, \dot{x}) \eta^{*i}_{p/} \overline{\eta}^j_{p/}. \tag{3.1}$$

Using TAYLORS's expansion, we have

$$g^*_{(ij)} = g_{(ij)} + \left(\frac{d g^*_{(ij)}}{ds} \right)_0 \delta s + \frac{1}{2} \left(\frac{d^2 g^*_{(ij)}}{ds^2} \right)_0 (\delta s)^2 + \cdots \tag{3.2a}$$

$$\eta^{*i}_{p/} = \eta^i_{p/} + \left(\frac{d \eta^{*i}_{p/}}{ds} \right)_0 \delta s + \frac{1}{2} \left(\frac{d^2 \eta^{*i}_{p/}}{ds^2} \right)_0 (\delta s)^2 + \cdots \tag{3.2b}$$

and

(3.2c) $$\overline{\eta}^i{}_{p/} = \eta^i{}_{p/} + \left(\frac{d\overline{\eta}^i{}_{p/}}{ds}\right)_0 (\delta s) + \frac{1}{2}\left(\frac{d^2\overline{\eta}^i{}_{p/}}{ds^2}\right)(\delta s)^2 + \cdots$$

where d/ds has been used for the operator

$$\left(\frac{\partial}{\partial x^k}\right)\left(\frac{dx^k}{ds}\right) + \frac{dx'^k}{ds}\frac{\partial}{\partial \bar{x}^k},$$

the element of support being understood to be in the direction of the tangent to C.

From FRENET's formulae established in the preceding section we have

(3.3) $$\frac{d\,\eta^{*i}{}_{p/}}{ds} = -k^*{}_{p-1}\,\eta^{*i}{}_{p-1/} + k^*{}_p\,\eta^{*i}{}_{p+1/} - \Gamma^{*i}{}_h\,\eta^{*h}{}_{p/}$$

where $\Gamma^i{}_h$ has deen used to indicate

$$\Gamma^i{}_{hk}\frac{dx^k}{ds} + C^i{}_{hk}\frac{dx^k}{ds},$$

for couvenience and star is used to indicate the value of a quantity at P^*.

Also, since $\overline{\eta}^i{}_{p/}$ is obtained from $\eta^i{}_{p/}$ by parallel displacement, we have

(3.4) $$\frac{d\overline{\eta}^i{}_{p/}}{ds} = -\Gamma^{*i}{}_h\,\eta^{*h}{}_{p/}.$$

From (1.5) we have

(3.5) $$\frac{d\,g^*{}_{(ij)}}{ds} = g^*{}_{(ih)}\,\Gamma^{*h}{}_j + g^*{}_{(hj)}\,\Gamma^{*h}{}_i.$$

Differentiating (3.3), and making use of it, we obtain

(3.6) $$\begin{aligned}\frac{d^2\eta^{*i}{}_{p/}}{ds^2} = &-\left(\frac{dk^*{}_{p-1}}{ds}\delta^i{}_h - \Gamma^{*i}{}_h\,k^*{}_{p-1}\right)\eta^{*h}{}_{p-1/} + \left(\frac{dk^*{}_p}{ds}\delta^i{}_h - \Gamma^{*i}{}_h\,k^*{}_p\right)\eta^{*h}{}_{p+1/}\\ &-\frac{d\Gamma^{*i}{}_h}{ds}\eta^{*h}{}_{p/} + k^*{}_{p-1}\,k^*{}_{p-2}\,\eta^{*i}{}_{p-2/} + k^*{}_p\,k^*{}_{p+1}\,\eta^{*i}{}_{p+2/}\\ &-(k^{*2}{}_{p-1} + k_p{}^{*2})\,\eta^{*i}{}_{p/}.\end{aligned}$$

Differentiating (3.4), and making use of it we obtain

(3.7) $$\frac{d^2\overline{\eta}^i{}_{p/}}{ds^2} = -\frac{d\,\Gamma^{*i}{}_h}{ds}\,\eta^{*h}{}_{p/} + \Gamma^{*i}{}_h\,\Gamma^{*h}{}_k\,\eta^{*k}{}_{p/}.$$

Differentiating (3.5), and making use of it, we obtain

(3.8) $$\begin{aligned}\frac{d^2 g^*{}_{(ij)}}{ds^2} = g^*{}_{ih}\frac{d\,\Gamma^{*h}{}_j}{ds} + g_{(hj)}\frac{d\,\Gamma_i{}^{*h}}{ds} + \Gamma^{*h}{}_j\,(\Gamma^*{}_{ih} + \Gamma^*{}_{hi})\\ + \Gamma^{*i}{}_i\,(\Gamma^*{}_{jh} + \Gamma^*{}_{hj})\end{aligned}$$

where

$$\Gamma^*_{ij} = g^*_{(hj)} \Gamma^{*h}_i . \tag{3.9}$$

From (3.1), (3.2a), (3.2b) and (3.2c) we obtain

$$\begin{aligned} \cos \delta \vartheta_p = 1 &+ \left[g_{(ij)} \left(\frac{d \eta^{*i}_{p/}}{ds} + \frac{d \overline{\eta}^i_{p/}}{ds} \right)_0 \eta^j_{p/} + \left(\frac{d g^*_{(ij)}}{ds} \right)_0 \eta^i_{p/} \eta^j_{p/} \right] \delta s \\ &+ \left[g_{(ij)} \left(\frac{d \eta^{*i}_{p/}}{ds} \right)_0 \left(\frac{d \overline{\eta}^j_{p/}}{ds} \right)_0 + \frac{1}{2} g_{(ij)} \left(\frac{d^2 \eta^{*i}_{p/}}{ds^2} + \frac{d^2 \overline{\eta}^i_{p/}}{ds^2} \right)_0 \eta^j_{p/} \right. \\ &+ \left. \left(\frac{d g_{(ij)}}{ds} \right)_0 \left(\frac{d \eta^{*i}_{p/}}{ds} + \frac{d \overline{\eta}^i_{p/}}{ds} \right)_0 \eta^j_{p/} + \frac{1}{2} \left(\frac{d^2 g^*_{(ij)}}{ds^2} \right)_0 \eta^i_{p/} \eta^h_{p/} \right] (\delta s)^2 \\ &+ \cdots\cdots\cdots \end{aligned} \tag{3.10}$$

Substituting the values of the various quantities from equations (3.3)—(3.8) in (3.10) and making use of the orthogonality relations (2.4), we obtain

$$\cos \delta \vartheta_p = 1 - \frac{1}{2} (k^2_{p-1} + k^2_p) (\delta s)^2 + \cdots$$

which may also be written as

$$\frac{1 - \cos \delta \vartheta_p}{(\delta \vartheta_p)^2} \left(\frac{\delta \vartheta_p}{\delta s} \right)^2 = \frac{1}{2} (k^2_{p-1} + k_p^2) + \text{terms containing second and higher powers of } \delta s.$$

Taking limits as $P^* \to P$, *i. e.* as $\delta s \to 0$, we obtain

$$\left(\frac{d\vartheta_p}{ds} \right)^2 = k^2_{p-1} + k_p^2 . \tag{3.11}$$

If ϱ_p be used for the pth radius of curvature, *i. e.* for the reciprocal of the pth curvature, then (3.11) can be written as

$$\left(\frac{d\vartheta_p}{ds} \right)^2 = \frac{1}{\varrho^2_{p-1}} + \frac{1}{\varrho_p^2} . \tag{3.12}$$

The corresponding result was established for FINSLER spaces by KAUL [5] and for generalised RIEMANN spaces by the author [2]. Both these results follow from (3.12) as particular cases.

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(Manuscript received November 20, 1962)

ÖZET

İkinci nev'i umumileştirilmiş Finsler uzayları [3] le incelenmiştir. Bu araştırmada ise n-boyutlu umumileştirilmiş Finsler uzayları için Frenet formülleri elde edilmiştir. Finsler uzayları ve umumileştirilmiş Riemann uzayları için Frenet formülleri ([1], [2]) birer hususî hal olarak elde edilmektedir. Umumileştirilmiş Riemann uzaylarında eğrilikler arasında yazar tarafından bulunan bir bağlantı umumileştirilmiş Finsler uzayları için de ispat edilmiştir. Kaul tarafından Finsler uzayları için bulunan benzer bir hassa [5] bunun hususî bir hali şeklinde elde edilmektedir.