ON NETWORKS OF MINIMUM CONSTRUCTION COST*

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In the present paper two similar procedures are given to determine the connection network of minimum construction cost when the set of terminals is specified. It is assumed that terminals are connected by straight links and the amount of flow between any pair of terminals is known. The building cost per unit length for a link is supposed to be the sum of a charge that is proportional to the total flow carried by this link and a constant fixed charge.

1. Introduction. The paper is concerned with finding the optimal connection network when the set of terminals is specified. Terminals are to be connected by straight links and the amount of flow between any pair of terminals is given. The building cost per unit length for a connection link is assumed to be the sum of a charge that is proportional to the total flow carried by this link and a constant fixed charge. If, in particular, the building cost consisted of the fixed charge only, the present problem would reduce to finding the shortest connection network which has already been treated by R. C. Prim [1]**. On the other hand, if the fixed charge was zero, the optimal connection network would be the complete network which contains all the possible links. Two similar procedures are given to find the solution to the general problem. The first procedure is described in detail; it consists of constructing a sequence of optimal networks starting from the shortest tree. The basic idea in the second procedure is the same as in the first, but it is the complete network that is chosen as the first element of the sequence. The first procedure is equally applicable to the problem of modifying a previously existing connection network when the amounts of flow between terminals increase in time at a constant ratio.

The practicality of these procedures depends on the ratio of the proportional charge per unit flow and the fixed charge. If the fixed charge prevailed,

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** Numbers in square brackets refer to the Bibliography at the end of the paper.

the first procedure could lead to the optimal network required at a relatively small number of steps. The second procedure is preferable if the fixed charge is small compared to the proportional charge per unit flow.

2. Formulation of the problem. The building cost per unit length of a connection link is assumed to be represented by the diagram of Fig. 1, which is characterized by two constants α and β. For a given connection network,

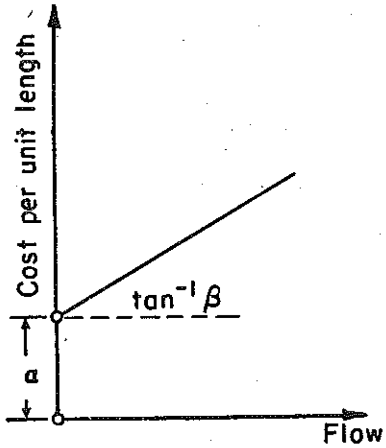

Fig. 1

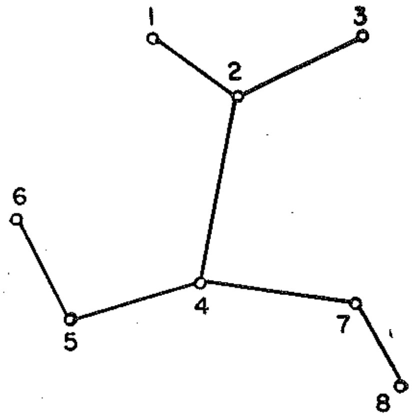

Fig. 2

the total building cost is then given by the following sum extended over all connection links

$$C = \Sigma L (\alpha + \beta Q), \tag{1}$$

where L is the length of a generic link and Q the total flow carried by this link. It is obvious that in a given network the flow between any two terminals must follow the shortest path available. In the following, this principle will be called the shortest path principle. Thus, for a given network and specified amounts of flow between each pair of terminals, the cost defined by Equation (1) is uniquely determined.

It will he convenient for the subsequent analysis to treat briefly the two extreme cases of either $\alpha = 0$ or $\beta = 0$. For $\alpha = 0$, the optimal network is the one that is obtained by directly connecting every pair of terminals between which there is a non-zero flow. If the flow associated with every pair of terminals is non-zero, the optimal solution is the complete network that contains ail possible links between terminals.

For $\beta = 0$, the optimal network is the shortest topological tree [1]. It is interesting to observe that the shortest tree remains the optimal network, whatever the values of α and β, if the flow between any two terminals that are not directly connected in the shortest tree is specified to be zero.

In general, the optimal solution of the problem formulated above is an ordinary connection network, i. e. neither a shortest tree nor the complete network, lying between these two extreme solutions. Two procedures similar to each other will be presented in the next section each one of which enable one to obtain the optimal network or networks in a straightforward step-by-step manner starting from either one or the other extreme solution.

3. Construction of the optimal network (first procedure). This procedure consists of constructing a sequence of networks in such a way that each network is optimal for a certain range of values of β and the totality of these ranges forms a closed interval on the β axis including $\beta = 0$ and $\beta = \beta^*$, β^* being the actual value of β. The first element of this sequence will be chosen as the suitable shortest tree optimal for $\beta = 0$. The subsequent elements of the sequence will be constructed one after another as β increases continuously starting from zero. The final network which is optimal for the range including β^* is, of course, the required optimal network. In this first procedure, α will be kept constant.

Before describing the procedure, it is convenient to introduce a few concepts which will be used in the following. Consider a number of links that could be inserted into a given network N. If the flow carried by any one of these links when it is inserted individually in N is the same as the flow carried by this link when all considered links are inserted simultaneously, these links will bo called indepedently insertable. In the shortest tree of Fig. 2, the links (1,3), (3,8), and (5,7) are independently insertable. According to Lemma 1 established in Appendix, the simultaneous insertion of such links becomes economical for a value of β which lies between the smallest and the largest of the β values associated with the individual insertions of these links.

If the flow carried by each one of a number of links when it is inserted individually is lessened when they are inserted simultaneously, these links will be called interdependent for insertion. In the shortest tree of Fig. 2, the links (1,5) and (1,6), for instance, are inter-dependent for insertion. According to Lemma 2, the simultaneous insertion of such links becomes economical for a value of β which cannot be smaller than the smallest of the β values associated with the individual insertion of these links, but can exceed the largest of them.

If the flow carried by each one of a number of links when it is inserted individually becomes larger when they are inserted simultaneously, these links will be called jointly insertable. In the shortest tree of Fig. 2, the links

(1.3) and (1.6) are jointly insertable. According to Lemma 3, the simultaneous insertion of such links becomes economical for a value of β which cannot exceed the largest of the β values associated with the individual insertions of these links, but can be less than the smallest of them.

From the three Lemmas established in Appendix, we can immediately deduce the following results. Suppose that a link available for insertion is not involved in any combination of jointly insertable links. We can then restrict ourselves to testing only the individual insertion of such a link according to Lemma 1 and 2. Otherwise, the joint insertions of this link in all possible combinations must be tested in addition to its individual insertion according to Lemma 3. Since the omission process is the reverse process to the insertion one, the preceding results must be reversed as far as omission is concerned. Namely, the joint omissions of a link in all possible combinations must be tested in addition to its individual omission according to Lemma 2 if this link is involved in combinations of links inter-dependent for re-insertion. Otherwise, we can restrict ourselves to testing its individual omission. Links inter-dependent for re-insertion will be called jointly omittable.

As remarked in the previous section, for $\beta = 0$ the optimal network is any one of the shortest trees (there may exist more than one shortest tree). For sufficiently small values of β, the shortest tree with the smallest sum $\Sigma L Q$ obviously remains the optimal network. The total cost of building for this network is expressed as

$$C^{(1)} = \alpha \Sigma L^{(1)} + \beta \Sigma L^{(1)} Q^{(1)} \tag{2}$$

where the superscripts (1) refer to the first element of the sequence. Since α is kept constant, $C^{(1)}$ is seen to be a linearly increasing function of β for this network. It is reasonable to expect that as β increases the modification of the shortest tree becomes necessary by inserting a new link or links and perhaps omitting a link or links. The insertion of new links introduces new shortest paths available for certain flows and therefore diminishes the flow-dependent part of the cost $\beta \Sigma L^{(1)} Q^{(1)}$ in Equation (2), although it augments the fixed charge $\alpha \Sigma L^{(1)}$. The probable omission of some links after the insertion process has been completed diminishes the fixed charge $\alpha \Sigma L^{(1)}$, whereas it augments the flow-dependent part of the cost. The β value for which the insertion of the link (p, q), for instance, first becomes economical is computed by an equation of the form

$$\beta_{pq} \sum_{m,n} \lambda_{mn} Q_{mn} - \alpha L_{pq} = 0 \tag{3}$$

where $\lambda_{mn} > 0$ is the difference between the lengths of the paths which the flow Q_{mn} follows before and after the insertion of this link, L_{pq} is the length of the link (p, q), and the sum is extended over all the flows that have changed their paths.

To obtain the second element of the sequence of optimal networks an insertion process first is applied to the shortest tree and the resulting network is modified by an omission process, if necessary. The insertion process is carried out by means of two sets of β values. The first set $\left\{\beta_{ij}^{(1)}\right\}$ is computed in such a way that for each element of it the individual insertion of the corresponding link in the shortest tree would first become economical. Each element of the second set of β values $\left\{\beta_{k}^{(1)}\right\}$ $(k=1, 2, \ldots, m)$ is associated with the simultaneous insertion of a number of jointly insertable links.

The insertion process is carried out according to the following principles :

1. The joint set $\left\{\beta_{ij}^{(1)}, \beta_{k}^{(1)}\right\}$ has only one element smaller than all others. The corresponding link or group of links has then to be inserted.

2. The joint set $\left\{\beta_{ij}^{(1)}, \beta_{k}^{(1)}\right\}$ has several equal elements that are smaller than all others. The corresponding links and/or groups of links have all to be inserted, provided that they are independently insertable.

3. If the set $\left\{\beta_{ij}^{(1)}, \beta_{k}^{(1)}\right\}$ attains its minimum for more than one element associated with links and/or groups of links inter-dependent for insertion, the link and/or group of links whose insertions result in the smallest sum $\Sigma L Q$ must be inserted. In the event that the set $\left\{\beta_{ij}^{(1)}, \beta_{k}^{(1)}\right\}$ attains its minimum for several groups of links such that the elements of each group are inter-dependent for insertion and/or for a number of groups of jointly insertable links such that certain combinations of these groups are inter-dependent for insertion, the process is applied to each group of links and/or to each combination of groups of jointly insertable links. In Fig. 2, the links (1,4), (1,5), and (1,6) on one hand, and the links (3, 4), (3, 7), and (2, 7) on the other hand constitute two such groups of links.

4. It may happen that the two preceding cases have to be combined. In Fig. 2, for example, if the insertions of the independently insertable links (1,3) and (4,8) and the links (1,4) and (2,6) that are inter-dependent for insertion first become economical, among the links (1,4) and (2,6) the one whose insertion results in the smallest sum $\Sigma L Q$ must be inserted in addition to inserting the links (1,3) and (4,8).

Now, the question arises whether or not it would be economical to take out some links after the insertion process has been completed. This could be the case because the insertion of new links is bound to diminish the amounts of flow along certain links, although the fixed charges (αL) associated with these links remain constant. Of course, the condition of connectedness must be respected in the omission process. The links which have to be omitted can be determined in a similar way. Consider the set $\left\{\bar{\beta}^{(1)}_{rs}\right\}$ whose generic element $\bar{\beta}^{(1)}_{rs}$ is computed for the re-insertion of the link (r, s) in the new network as if this link were not involved in it. In addition to the individual omissions of the links available, the joint omissions of the links that are inter-dependent for re-insertion must be considered. Because the re-insertions of such links might become economical for a value of β which exceeds the β values associated with the individual re-insertions of these links. Denote by $\left\{\bar{\beta}^{(1)}_{l}\right\}$ the set of these β values. Of course, only the omissions of the links along which amounts of flow have been diminished after the insertion process must be tested. If the condition

$$\max\left\{\bar{\beta}^{(1)}_{rs},\ \bar{\beta}^{(1)}_{l}\right\} \leq \min\left\{\beta^{(1)}_{ij},\ \beta^{(1)}_{k}\right\} \tag{4}$$

Is fulfilled, no link need he omitted. In this case, the shortest tree used for getting started ceases being optimal for $\beta = \beta_1$, where β_1 is given by

$$\beta_1 = \min\left\{\beta^{(1)}_{ij},\ \beta^{(1)}_{k}\right\}, \qquad (i \neq j) \tag{5}$$

and the network obtained by the insertion process becomes optimal for values of β slightly exceeding β_1.

Now, suppose that the condition (4) is violated by some elements of the set $\left\{\bar{\beta}^{(1)}_{rs},\ \bar{\beta}^{(1)}_{l}\right\}$. In this case, the insertion of new links must be supplemented by the omission of some links previously existing to obtain the subsequent optimal network. The omission process will be carried out as follows :

1. The links and/or groups of links with $\bar{\beta}^{(1)}_{rs}$ and/or $\bar{\beta}^{(1)}_{l}$ violating the condition (4) have all to he omitted, provided that they are independently omittable.

2. Among the links and/or groups of links inter-dependent for omission with $\bar{\beta}^{(1)}_{rs}$ and/or $\bar{\beta}^{(1)}_{l}$ violating the condition (4), the ones whose omissions first become economical must be omitted. If there exist several groups of inter-dependent links available for omission, each group cannot be handled, in general, separately as in the insertion process. On the contrary, the link and/or the group of links whose omissions first become economical have to be omitted.

3. It might happen that the two preceding cases have to he combined.

It is evident from the three Lemmas established in Appendix and the procedure itself that the network obtained in this way will be the optimal network as soon as the preceding one ceases to be optimal. In this case, β_1 is no longer equal to min $\beta_{ij}^{(1)}$, but is obtained by equating the total costs associated with these two networks.

Applying the insertion process and the omission process, if necessary, successively, the subsequent elements of the sequence of optimal networks can easily be obtained. The i-th element of this sequence is optimal for the values of β satisfying $\beta_{i-1} \leqq \beta \leqq \beta_i$. Suppose that we have $\beta_{n-1} \leqq \beta^* \leqq \beta_n$. The n-th element of the sequence is then the optimal network required.

Geometrically, the point with coordinates β and $C_{opt.}$ in the (β, C)-plane moves on the polygonal contour $A_0 A_1 A_2 \dots$ in Fig. 3 as β increases, where $C_{opt.}$ is the cost associated with the network optimal for the appropriate

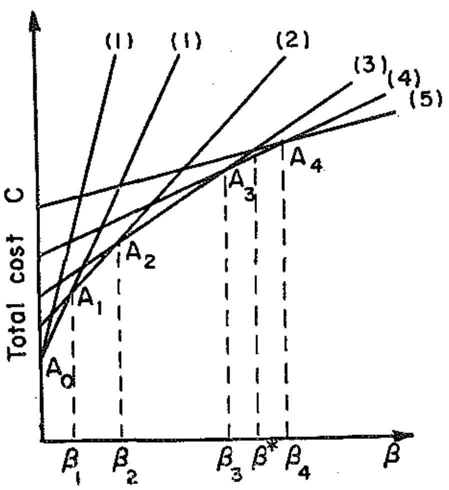

Fig. 3

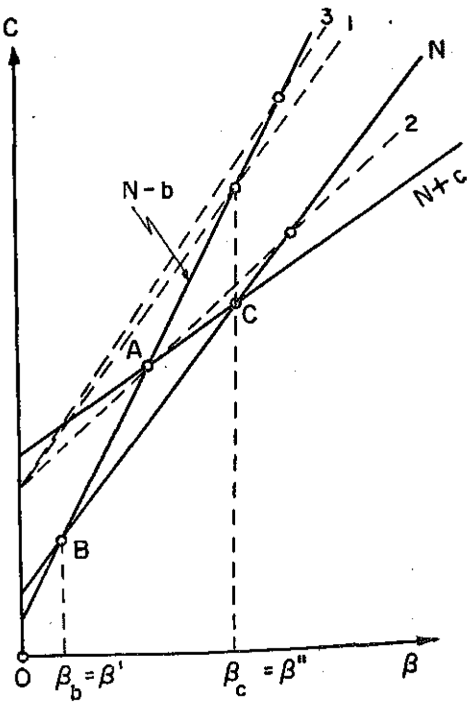

Fig. 4

value of β. The vertices of this contour correspond to the β values for which two successive elements of the sequence of optimal networks have equal costs. In Fig. 3, only the cost diagrams of optimal networks have been drawn.

In the procedure described above, it might not be necessary to construct all the elements of the sequence of optimal networks to obtain the network optimal for $\beta=\beta^*$. In fact, consider the k-th network optimal for $\beta_{k-1} \leqq \beta \leqq \beta_k$. If

$$\max \left\{\bar{\beta}_{rs}^{(k)},\ \bar{\beta}_{n}^{(k)}\right\} \leq \min \left\{\beta_{ij}^{(k)},\ \beta_{m}^{(k)}\right\}, \tag{6}$$

no link need be omitted from the network obtained by inserting a new link or links in the k-th one. This new network (the $(k+1)$-th one in this case) will, therefore, be the successive optimal network until it becomes economical to modify it. In this case, we merely have to continue the procedure. Now, suppose that the condition (6) is not fulfilled, but, instead, we have

$$\max \left\{\bar{\beta}_{rs}^{(k)},\ \bar{\beta}_{n}^{(k)}\right\} \leq \min \left\{\beta_{ij}^{(k+l)},\ \beta_{m}^{(k+l)}\right\} \tag{7}$$

where the superscript $(k+l)$ refers to the network obtained from the k-th network by performing the necessary insertion process $(l > 1)$. Since the $(k+l)$-th network is optimal for β values satisfying

$$\max \left\{\bar{\beta}_{rs}^{(k)},\ \bar{\beta}_{n}^{(k)}\right\} \leq \beta \leq \min \left\{\beta_{ij}^{(k+l)},\ \beta_{m}^{(k+l)}\right\} \tag{8}$$

according to the Theorem in Appendix, the procedure can be continued by applying a new insertion process to the $(k+l)$-th network without applying the necessary omission process to it. Here, l is the number of the optimal networks by-passed. It should he noted that the necessary omission process must be carried out if the conditions (6) and (7) are not fulfilled simultaueously.

In the beginning of this section, we had left undecided which shortest tree must be taken in getting started if there exist more than one shortest tree having equal sums $\Sigma L Q$ smaller than the analogous sums associated with all other shortest trees. It is obvious from the procedure that the one which first ceases being optimal must be taken.

Another ambiguity similar to the one described in the last paragraph arises in the insertion process if the set $\{\beta_{ij},\ \beta_k\}$ attains its minimum for more than one element associated with inter-dependent links the insertions of which result in equal sums $\Sigma L Q$ smaller than all the analogous sums corresponding to the insertions of other inter-dependent links. The insertion of each one of such links gives a different network, but the costs for these networks are represented by the same line in (β, C)-plane (see Fig. 3). Among these networks, the one which first ceases being optimal must be chosen as the subsequent optimal network if no link need be omitted from any one of these networks. Otherwise, omission process must be applied to each one of these networks and the resulting networks must be compared. The one whose cost first becomes equal to the cost of the preceding network must be chosen as the subsequent optimal network.

In applying the procedure, it may happen that there exist more than one shortest path available for a flow. In this case, any one of the shortest paths available can be used to carry the flow considered until one of them is first shortened by inserting a new link. The ambiguity is then removed. It should be noted that in computing the values of β associated with the insertions of new links in these shortest paths each new link must be supposed to carry the flow considered in addition to others. If the insertions of several new links in these shortest paths first become economical for the same value of β, the longest link must be inserted.

Second procedure. In this procedure, the complete network optimal for $\alpha=0$ is taken for getting started instead of the shortest tree in the first procedure, and α is increased starting from zero until it reaches α^*, the actual value of α, whereas β is kept constant throughout. A sequence of optimal networks is constructed in the same way as before to obtain the network optimal for $\alpha=\alpha^*$. If the total costs of the elements of this sequence are plotted against α, a figure similar to Fig. 3 is obtained. It is evident that the role played by the successive insertion processes applied in the first procedure is played by the successive omission processes applied in the second procedure, and vice-versa.

Remark. In the problem treated above, it was assnmed that the rate of flow associated with any pair of terminals did not depend on time. Now, suppose that rates of flow depend on time as follows :

$$Q_{ij}=f(t)\,Q_{ij}^{(0)}, \qquad (i\neq j) \tag{9}$$

where $f(t)$ is a monotonically increasing function of time and $Q_{ij}^{(0)}$ are constants. The resulting problem can be treated along the same lines as before substituting $\gamma=\beta f(t)$, where $f(t)$ is a given function. As $f(t)$ increases, the previously constructed optimal network has to be modified by inserting new links. Of course, it would not be economical to omit any link previously constructed. In particular, the optimal network at any given time can be obtained by inserting the links economical for the given value of $f(t)$ in the optimal network corresponding to $f(t)=1$.

4. Numerical example. Consider the set of the five terminals of Fig. 5 located on a distance-true map. The distances and the amounts of flow between these terminals are given by the two matrices of Fig. 6. Assuming without loss of generality that $\alpha=1$, we propose to find the network optimal for $\beta^*=4/7$ applying the first procedure. In this example, there is only one shortest tree which is drawn in full line in Fig. 5. To obtain the successive optimal networks, we compute the first set of β values :

$$\beta_{13}^{(1)}=\frac{5}{21},\quad \beta_{14}^{(1)}=\frac{11}{15},\quad \beta_{15}^{(1)}=\frac{20}{39},\quad \beta_{24}^{(1)}=\frac{15}{14},\quad \beta_{25}^{(1)}=\frac{17}{121},\quad \beta_{35}^{(1)}=\frac{5}{42}. \tag{10}$$

Since there are no links jointly insertable in this shortest tree, only the β values associated with the individual insertions of the links have been computed. The smallest element of this first set is $\beta_{35}^{(1)}$. The link (3,5) must

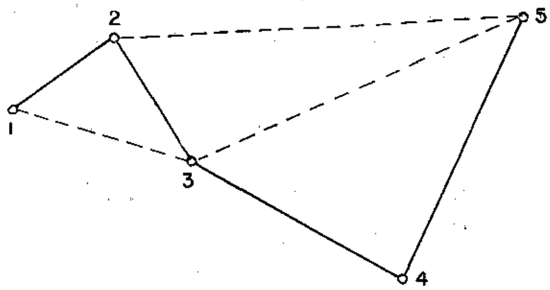

Fig. 5

then be inserted. Now, the individual omissions of the links (3,4) and (4,5) from the resulting network must be tested on one hand, the insertions of all the links available for insertion in the resulting network must be tested on the other hand. The re-insertions of the links (3,4) and (4,5) become economical for

$$\bar{\beta}_{34}^{(1)} = \frac{5}{68}, \qquad \bar{\beta}_{45}^{(1)} = \frac{2}{13}. \tag{11}$$

	1	2	3	4	5
1		1	1.5	3.3	4
2	1		1.2	3	3.4
3	1.5	1.2		2	3
4	3.3	3	2		2.4
5	4	3.4	3	2.4	

Distance Matrix

	1	2	3	4	5
1		1	4	2	3
2	1		0.5	1	8
3	4	0.5		5	7
4	2	1	5		6
5	3	8	7	6	

Flow Matrix

Fig. 6

On the other hand, we have

$$\beta_{13}^{(2)}=\frac{5}{21}, \quad \beta_{14}^{(2)}=\frac{11}{6}, \quad \beta_{15}^{(2)}=\frac{10}{9}, \quad \beta_{24}^{(2)}=5, \quad \beta_{25}^{(2)}=\frac{17}{44}. \tag{12}$$

Since max $\bar{\beta}_{rs}^{(1)}=\bar{\beta}_{45}^{(1)}<$ min $\beta_{ij}^{(2)}=\beta_{13}^{(2)}$, it is not necessary to take out the link (4,5) for continuing the procedure according to the Theorem in Appendix. Instead, the link (1,3) must be inserted. In reality, the second element of the sequence of optimal networks is the network obtained from the shortest tree by inserting the link (3,5) and omitting the link (4,5) simultaneously. However, this network has been by-passed to shorten the procedure. Now, we easily compute for the reinsertions of the links (1,2) and (2,3)

$$\bar{\beta}_{12}^{(2)}=\frac{10}{17}, \qquad \bar{\beta}_{23}^{(2)}=\frac{120}{1235} \tag{13}$$

on one hand, and for the insertions of the available links

$$\beta_{14}'=\frac{33}{4}, \qquad \beta_{15}'=\frac{8}{3}, \qquad \beta_{24}'=15, \qquad \beta_{25}'=\frac{34}{67} \tag{14}$$

on the other hand. Since max $\bar{\beta}_{rs}^{(2)}=\bar{\beta}_{12}^{(2)}>$ min $\beta_{ij}'=\beta_{25}'$, the theorem previously mentioned does not apply. The link (1,2) must therefore be omitted. We compute for the resulting network

$$\beta_{12}^{(3)}=\frac{10}{17}, \quad \beta_{14}^{(3)}=\frac{33}{4}, \quad \beta_{15}^{(3)}=\frac{8}{3}, \quad \beta_{24}^{(3)}=15, \quad \beta_{25}^{(3)}=\frac{17}{32}, \quad \beta_{1}^{(3)}=\frac{11}{21}, \tag{15}$$

where $\beta_1^{(3)}$ is the value of β for which the simultaneous insertion of the links (1,2) and (2,5) becomes economical. Thus, the links (1,2) and (2,5) must be inserted simultaneously. Now, the individual omissions of the links (1,3), (2,3), (3,5) and the simultaneous omission of the links (1,3) and (2,3) must be checked. We compute

$$\bar{\beta}_{13}^{(3)}=\frac{5}{14}, \qquad \bar{\beta}_{23}^{(3)}=\frac{8}{13}, \qquad \bar{\beta}_{35}^{(3)}=\frac{15}{49}, \qquad \bar{\beta}_{1}^{(3)}=\frac{9}{128}, \tag{16}$$

where $\bar{\beta}_1^{(3)}$ is the β value for which the simultaneous re-insertion of the links (1,3) and (2,3) becomes economical. Since $\bar{\beta}_{23}^{(3)}>\beta^*>\beta_1^{(3)}$, the link (2,3) must be omitted. To check, we compute for the resulting network

$$\beta_{14}^{(4)}=\frac{33}{4}, \qquad \beta_{15}^{(4)}=\frac{10}{3}, \qquad \beta_{23}^{(4)}=\frac{8}{12}, \qquad \beta_{24}^{(4)}=1. \tag{17}$$

Since these values are larger than $\beta^*=4/7$, no link need be inserted. Thus, the network obtained by omitting the link (2,3) is the required optimal network (see Fig. 7).

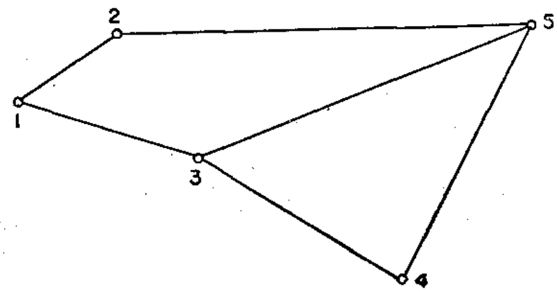

Fig. 7

APPENDIX

The three Lemmas and the Theorem to which reference has been made in Section 3 will he established here.

Lemma 1. The simultaneous insertion of a number of indepedently insertable links in a network N cannot precede the individual insertion of the link whose insertion first becomes economical, or, succeed the individual insertion of the link whose insertion last becomes economical.

Proof. For the moment, let us take two links (p, q) and (r, s). The β values for the individual insertions and the simultaneous insertion of these links are computed from the following equations (sec Equation 3 in Section 3)

$$\beta_{pq} \sum_{k,l} \lambda_{kl} Q_{kl} - \alpha L_{pq} = 0, \qquad \beta_{rs} \sum_{m,n} \lambda_{mn} Q_{mn} - \alpha L_{rs} = 0$$

and

$$\beta_{pq,\,rs} \Big[\sum_{k,l} \lambda_{kl} Q_{kl} + \sum_{m,n} \lambda_{mn} Q_{mn} \Big] - \alpha (L_{pq} + L_{rs}) = 0.$$

Suppose $\beta_{pq} < \beta_{rs}$. It can readily be verified that $\beta_{pq} < \beta_{pq,\,rs} < \beta_{rs}$. Similarly, the value $\beta_{pq,\,rs,\,tz}$ for which the simultaneous insertion of the links (p, q), (r, s) and (t, z) becomes economical lies between β_{tz} and $\beta_{pq,\,rs}$, etc.

Lemma 2. The simultaneous insertion of a number of links inter-dependent for insertion cannot precede the individual insertion of the link whose insertion first becomes economical, but can succeed the individual insertion of the link whose insertion last becomes economical.

Lemma 3. The simultaneous insertion of a number of jointly insertable links cannot succeed the individual insertion of the link whose insertion last becomes economical, but can precede the individual insertion of the link whose insertion first becomes economical.

These two Lemmas are proved in the same manner as in Lemma 1.

Theorem. Let $\{\beta_{ij}, \beta_k\}$ be the set of β values associated with the individual insertions of all the links available for insertion and the simultaneous insertions of the jointly insertable links in a given network N. Similarly, let $\{\bar{\beta}_{rs}, \bar{\beta}_l\}$ be the set of β values associated with the individual omissions of the existing links and the simultaneous omissions of the jointly omittable links from N. If

$$\beta' = \max\{\bar{\beta}_{rs}, \bar{\beta}_l\} \leq \beta'' = \min\{\beta_{ij}, \beta_k\},$$

then the network N is optimal for β values satisfying $\beta' \leq \beta \leq \beta''$.

Proof. Denote by $N+c$ and $N-b$ the two networks obtained from N by inserting either the link c or the jointly insertable links c in N, and omitting either the link b or the jointly omittable links b from N. If b and c are independently insertable in $N-b$, the insertion of c in $N-b$ becomes economical for the same value of β as in N. The cost for the network $N-b+c$ in this case is represented by the line labelled 1 in Fig. 4. The costs for this network are represented by the lines 2 and 3 respectively according to whether b and c are inter-dependent for insertion or jointly insertable in $N-b$. It is seen from Fig. 4 that in either case the cost line for $N-b+c$ cannot lie below the cost line for N in the interval (β_b, β_c). Now, suppose that $\beta' = \beta_b$ and $\beta'' = \beta_c$. Since the cost line for $N-b_1-\cdots-b_m$ intersects the cost line for N at a value of β smaller than β_b according to Lemma 2, and since the cost line for $N+c_1+\cdots+c_n$ intersects the cost line for N at a value of β larger than β_c according to Lemma 3, the cost line for the network $N-b_1-\cdots-b_m+c_1+\cdots+c_n$ cannot lie below the cost line for N in the interval (β', β''). This proves the theorem.

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İSTANBUL ÜNİVERSİTESİ
MATEMATİK ENSTİTÜSÜ
İSTANBUL, TÜRKİYE

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ÖZET

Bu makalede terminaller cümlesi verildiği takdirde inşa masrafı minimum olan irtibat şebekesinin tâyini için benzer iki metod verilmektedir. Terminallerin doğrusal irtibat halkaları ile bağlandığı ve herhangi iki terminal arasındaki akım miktarının bilindiği kabul edilmektedir. Bir irtibat halkasının birim uzunluğu için inşa masrafının bu halkanın taşıdığı akım ile orantılı bir masraf ile sabit bir masrafın toplamına eşit olduğu farzedilmektedir.