ON THE MAKSIMUM TERM, ORDER AND TYPE OF THE FUNCTION DEFINED BY THE SERIES

$$\sum_{1}^{\infty} a_n e^{\lambda_n \psi(x)}$$

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Abstract

In this paper the growth of an indefinitely increasing function of a real variable has been studied in relation to another function by introducing the notion of order and type. An attempt has been made here to unify «various aspects» of the two theories of entire functions defined by TAYLOR series and DIRICHLET series respectively, which have so far been treated separately by different workers in the two fields. Some applications given in section 5 are intended to emphasize this fact.

1. Consider the function

$$F(x) = \sum_{1}^{\infty} a_n e^{\lambda_n \cdot \psi(x)} \tag{1.1}$$

where,

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty \quad , \tag{1.2}$$

$\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim\limits_{n\to\infty} \lambda_n = \infty$; $\{a_n\}$ $(n = 1,2, \ldots\ldots)$ is a sequence of real positive numbers and $\psi(x)$ is an increasing positive function of a real variable x satisfying the following conditions :

(1.3) (i) $\psi(x)$ tends to infinity as $x \to \infty$.

(ii) $\psi(x)$ has an inverse, that is, if $y = \psi(x)$, then there exists a function ψ^{-1} such that $\psi^{-1}(y) = x$.

(iii) $\psi(x+k) - \psi(x) = \Phi(x) = O(1)$.

Also, we shall suppose throughout that $\Phi(x)$ is a positive bounded function of real variable x.

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In this paper we study the order and type of the function defined by (1.1). Here we have attempted to unify various aspects of the two theories of entire functions defined by TAYLOR series and DIRICHLET's series respectively, which have so far been treated separately by different workers in the two fields. Applications given in section 5 are intended to emphasize this fact.

2. Theorem 1.

If $\mu(x, F)$ and $\nu(x, F)$ denote the maximum term and the rank of the maximum term respectively of the series,

$$F(x) = \sum_{1}^{\infty} a_n e^{\lambda_n \cdot \psi(x)}$$

and $\psi'(x)$, the differential coefficient of $\psi(x)$, is finite everywhere and integrable then, for every $x \geq t \geq x_0$,

$$\log \mu(x, F) = \log \mu(x_0, F) + \int_{x_0}^{x} \lambda_{\nu(t, F)} . \psi'(t)\, dt. \tag{2.1}$$

Proof:

By hypothesis we have,

$$\mu(x, F) = a_{\nu(x, F)} . e^{\lambda_{\nu(x, F)} . \psi(x)} \tag{2.2}$$

where $a_{\nu(x, F)}$ and $\lambda_{\nu(x, F)}$ are constants in intervals, have an enumerable number of discontinuities and change values at these discontinuities only, therefore $a_{\nu(x, F)}$ and $\lambda_{\nu(x, F)}$ (and hence $\mu(x, F)$ are differentiable outside a set of measure zero and the differential coefficients of $a_{\nu(x, F)}$ and $\lambda_{\nu(x, F)}$ vanish almost everywhere. Thus we have,

$$\begin{aligned}\mu'(x, F) &= a_{\nu(x, F)} \cdot e^{\lambda_{\nu(x, F)}} \cdot \psi(x) \cdot \lambda_{\nu(x, F)} \cdot \psi'(x)\\ &= \mu(x, F) \cdot \lambda_{\nu(x, F)} \cdot \psi'(x)\end{aligned}$$

almost everywhere.

On integration we have,

$$\begin{aligned}\log \mu(x, F) - \log \mu(x_0, F) &= \int_{x_0}^{x} \frac{\mu'(t, F)}{\mu(t, F)}\, dt\\ &= \int_{x_0}^{x} \lambda_{\nu(t, F)} \cdot \psi'(t)\, dt.\end{aligned}$$

Lemma 1.

$\log \mu(x, F)$ is a convex function of $\psi(x)$.

In view of the fact that $\lambda_{\nu(x, F)}$ and $\psi(x)$ are both positive and $\lambda_{\nu(x, F)}$ is nondecreasing also, the lemma follows from theorem 1.

Lemma 2. *If,*

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0$$

then,

$$\log F(x) \sim \log \mu(x, F) \quad \text{as} \quad x \to \infty. \tag{2.3}$$

Proof:

It is evident that

$$\frac{\log F(x)}{\log \mu(x, F)} > 1. \tag{2.4}$$

On the other hand from (1.3, iii) we have, for $\varepsilon > 0$

$$\psi(x+\varepsilon) - \psi(x) = \Phi(x) \tag{2.5}$$

where $\Phi(x)$ is a positive bounded function.

Now, for $0 < \varepsilon' < \Phi(x)$, we can choose a positive integer $N(\varepsilon)$ such that

$$\frac{\log n}{\lambda_n} < \varepsilon' \qquad \text{for } n > N(\varepsilon).$$

Therefore, in view of (2.5) we have,

$$F(x) = \sum_{1}^{N(\varepsilon)} a_n e^{\lambda_n \cdot \psi(x)} + \sum_{N(\varepsilon)+1}^{\infty} [\, a_n e^{\lambda_n \cdot \psi(x+\varepsilon)} \cdot e^{-\lambda_n \cdot \Phi(x)} \,]$$

$$< N(\varepsilon) \cdot \mu(x, F) + \mu(x+\varepsilon, F) \sum_{N(\varepsilon)+1}^{\infty} \frac{1}{n^{\frac{\Phi(x)}{\varepsilon'}}}$$

$$< K \mu(x+\varepsilon, F) \qquad [\, \Phi(x) > \varepsilon' \,]$$

where K is a constant depending on $F(x)$ and on ε'.

Therefore,

(2.6) $$\log F(x) < \log \mu(x+\varepsilon, F) + O(1).$$

If the series converges for $x \leq \infty$, $\log \mu(x, F)$ being convex, increases indefinitely with $\psi(x)$ which tends to infinity as $x \to \infty$.

Hence in view of (2.4) and (2.6) we have,

$$\lim_{x\to\infty} \frac{\log F(x)}{\log \mu(x, F)} = 1.$$

Thus

$$\log F(x) \sim \log \mu(x, F)$$

3. Let

(3.1) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log F(x)}{\psi(x)} = \begin{matrix}\varrho\\ \lambda\end{matrix}, (0 \leq \lambda \leq \varrho < \infty).$$

We shall refer to the constants ϱ and λ as defined in (3.1) by ψ—order and lower ψ—order respectively of the function $F(x)$ as defined in (1.1) which will be said to be of «regular ψ—order» when $\varrho = \lambda$. The justification for this lies in the fact that ϱ and λ depend on the function $\psi(x)$.

Remark:

In view of lemma 2, proved above in section two, we have,

(3.2) $$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(x, F)}{\psi(x)} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log F(x)}{\psi(x)} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Teorem 2.

The necessary and sufficient condition that

$$F(x) = \sum_{1}^{\infty} a_n e^{\lambda_n \cdot \psi(x)}$$

should be an integral function of finite ψ—order ϱ, is that

(3.3) $$\lim_{n\to\infty} \inf \frac{\log (a_n)^{-1}}{\lambda_n \log \lambda_n} = \frac{1}{\varrho}$$

provided that

(3.4) $$\lim_{n\to\infty} \sup \frac{n}{\lambda_n} = D < \infty.$$

Proof:

Let us suppose that $F(x)$ is of finite ψ—order ϱ. Then, from the definition of ψ—order (3.1) we have,

$$F(x) < \exp[\exp(k.\psi(x))] \quad , \quad (k > \varrho).$$

Hence,

$$a_n < \exp[\exp(k.\psi(x)) - \lambda_n.\psi(x)].$$

The expression on the right hand side of this inequality attains its maximum value $\left[\frac{ek}{\lambda_n}\right]^{\lambda_n/k}$ when,

$$\exp[k.\varphi(x)] = \lambda_n/k.$$

Hence,

$$a_n < \left[\frac{e.k}{\lambda_n}\right]^{\lambda_n/k}.$$

Therefore,

$$\lim_{n\to\infty} \inf \frac{\log (a_n)^{-1}}{\lambda_n \log \lambda_n} \geq \frac{1}{k}.$$

Thus we observe that

$$\lim_{n\to\infty} \inf \frac{\log (a_n)^{-1}}{\lambda_n \log \lambda_n}$$

is not negative. Now, let us set

$$\lim_{n\to\infty} \inf \frac{\log (a_n)^{-1}}{\lambda_n . \log \lambda_n} = \mu$$

where μ is zero, positive or infinite. Therefore, for any $\varepsilon < 0$, we have,

(3.5) $$a_n < \lambda_n^{-(\mu-\varepsilon)\lambda_n} \qquad (0 < \varepsilon < \mu)$$

for all indices $n > n_0$.

Therefore, in view of (3.4) we get,

$$(a_n)^{1/n} < \lambda_n^{\left[\frac{\mu-\varepsilon}{D+\varepsilon'}\right]} \qquad (\varepsilon' > 0);\ \left\{(\mu-\varepsilon) > (D+\varepsilon')\right\}.$$

Thus $(a_n)^{1/n}$ tends to zero as n tends to ∞ provided $\mu > 0$, since $(\mu-\varepsilon)$ is positive and λ_n tends to infinity as $n \to \infty$, hence $F(x)$ is an integral function.

Now, if μ is finite, then

$$F(x) = \sum_{1}^{n_0} a_n e^{\lambda_n . \psi(x)} + \sum_{n_0+1}^{\infty} a_n e^{\lambda_n . \psi(x)}$$

$$< Ae^{\lambda_{n_0} \cdot \psi(x)} + \sum_{n_0+1}^{\infty} \left[\frac{e^{\psi(x)}}{\lambda_n^{\mu-\varepsilon}}\right]^{\lambda_n}$$

where A is a constant.

Choose an integer M such that

$$\lambda_M \leq 2^{1/(\mu-\varepsilon)} \cdot \exp\left[\frac{\psi(x)}{\mu-\varepsilon}\right] < \lambda_{M+1} .$$

When $\lambda_{n_0+1} \leq \lambda_n \leq \lambda_M$ we have,

$$\sum_{n_0+1}^{M} \left[\frac{e^{\mu(x)}}{\lambda_n^{\mu-\varepsilon}}\right]^{\lambda_n} \leq \exp\left[\psi(x) \cdot 2^{1/(\mu-\varepsilon)} \cdot \exp\frac{\psi(x)}{\mu-\varepsilon}\right] \sum_{n_0+1}^{M} [\lambda_n^{-\lambda_n(\mu-\varepsilon)}]$$

$$= B \exp\left[\psi(x) \cdot 2^{1/(\mu-\varphi)}, \exp\left\{\frac{\psi(x)}{\mu-\varepsilon}\right\}\right]$$

where B is a constant independent of x, since the series $\sum_{n_0+1}^{M} \lambda_n^{-\lambda_n(\mu-\varepsilon)}$ in view of (3.4), is convergent if extended to ∞.

Also

$$\sum_{M+1}^{\infty} \left[\frac{e^{\psi(x)}}{\lambda_n^{\mu-\varepsilon}}\right]^{\lambda_n} < \sum_{M+1}^{\infty} \left[\frac{e^{\psi(x)}}{\lambda_{M+1}^{\mu-\varepsilon}}\right]^{\lambda^n} < \sum_{M+1}^{\infty} (1/2)^{\lambda_n} = \text{constant}.$$

Thus

$$F(x) < B \exp\left[\psi(x) \cdot 2^{1/(\mu-\varepsilon)}, \exp\left\{\frac{\psi(x)}{\mu-\varepsilon}\right\}\right] + 0(1)$$

for all large values of $\psi(x)$.

Therefore

$$\varrho = \lim_{x\to\infty} \sup \frac{\log\log F(x)}{\psi(x)} \leq \lim_{x\to\infty} \sup \left[\frac{\log \psi(x) + \frac{1}{\mu-\varepsilon}\log 2 + \frac{\varphi(x)}{\mu-\varepsilon}}{\psi(x)}\right]$$

(3.6) $$\text{or } \varrho \leq \frac{1}{\mu-\varepsilon} .$$

If $\mu = \infty$, the argument, with an arbitrary large number instead of μ shows that $\varrho = 0$.

Again, there exists an infinite number of positive integers such that

$$a_n > \lambda_n^{-(\mu+\varepsilon)\cdot\lambda_n}$$

or

$$a_n e^{\lambda_g \cdot \psi(x)} > [\, e^{\psi(x)} \,.\, \lambda_n^{-(\mu+\varepsilon)} \,]^{\lambda_n} .$$

Hence, if we take $e^{\psi(x)} = [\, 2 \,.\, \lambda_n \,]^{\mu+\varepsilon}$, then

$$F(x) > a_n e^{\lambda_n \cdot \psi(x)} > 2^{\lambda_n \cdot (\mu+\varepsilon)}$$

$$= \exp\left[\frac{\mu+\varepsilon}{2} \cdot \exp\frac{\psi(x)}{\mu+\varepsilon} \cdot \log 2\right]$$

for sufficiently large values of $\psi(x)$.

Therefore,

$$\varrho = \lim_{x\to\infty} \sup \frac{\log\log F(x)}{\psi(x)} \geq \lim_{x\to\infty} \sup \left[\frac{\log(\mu+\varepsilon) - \log 2 + \frac{\psi(x)}{\mu+\varepsilon} + \log\log 2}{\psi(x)}\right]$$

or

$$(3.7) \qquad \varrho \geq \frac{1}{\mu+\varepsilon} .$$

If $\mu = 0$, then $F(x)$ is of finite order.

Thus, the result in (3.3) follows from (3.6) and (3.7) making ε tend to zero.

4. A better estimate of the growth of the function $F(x)$ in relation to the function $\psi(x)$ is obtained if we consider the limit of $\frac{\log F(x)}{e^{\varrho \cdot \psi(x)}}$. Thus let,

$$(4.1) \qquad \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log F(x)}{\exp[\varrho \cdot \psi(x)]} = \begin{matrix}T\\ t\end{matrix} , \qquad (0 \leq t \leq T \leq \infty)$$

where $\varrho\,(0 < \varrho < \infty)$ is the ψ—order of the function $F(x)$. We define «T» to be the ψ—type and «t» the lower ψ—type of the function $F(x)$ of ψ—order ϱ $(0 < \varrho < \infty)$ and in case, the limit in (4.1) exists i.e. $T = t$, we say that $F(x)$ is of perfectly ψ—regular growth.

Theorem 3.

The necessary and sufficient condition that the function

$$F(x) = \sum_{1}^{\infty} a_n e^{\lambda_n \cdot \psi(x)}$$

to be of type T of finite ψ—order $\varrho > 0$, is that

(4.2) $$\lim_{n\to\infty} \sup \frac{\lambda_n}{e\varrho} (a_n)^{\varrho/\lambda_n} = T.$$

Proof :

Let us set,

$$\lim_{n\to\infty} \sup \frac{\lambda_n}{e\varrho} (a_n)^{\varrho/\lambda_n} = \nu.$$

Suppose that $0 \leq \nu < \infty$. Let $\varepsilon > 0$, we have,

(4.3) $$a_n < \left[\frac{(\nu+\varepsilon)\, e\varrho}{\lambda_n}\right]^{\lambda_n/\varrho}$$

for $n > n_0$.

Hence,

$$F(x) = \sum_{1}^{n_0} a_n e^{\lambda_n \cdot \psi(x)} + \sum_{n_0+1}^{\infty} a_n e^{\lambda_n \cdot \psi(x)}$$

$$< A\, e^{\lambda_{n_0} \cdot \psi(x)} + \sum_{n_0+1}^{\infty} \left[\frac{(\nu+\varepsilon)\, e\, \varrho \,.\, e^{\varrho \cdot \psi(x)}}{\lambda_n}\right]^{\lambda_n/\varrho}.$$

The general term on the right hand side does not exceed its maximum which is $\exp[(\nu+\varepsilon)\,.\,e^{\varrho \cdot \psi(x)}]$ attained for $\lambda n = (\nu+\varepsilon)\,\varrho\,.\,e^{\varrho \cdot \psi(x)}$. Now choose an integer M such that

$$\lambda_M \leq (\nu + 2\varepsilon)\, e \,.\, \varrho \,.\, e^{\varrho \cdot \psi(x)} < \lambda_{M+1}.$$

When $\lambda_{n+1} \leq \lambda_n \leq \lambda_M$, we have,

$$\sum_{n_0+1}^{M} \left[\frac{(\nu+\varepsilon)\, e\, \varrho}{\lambda_n} \cdot e^{\varrho \cdot \psi(x)}\right]^{\lambda_n/\varrho} \leq [\text{Number of terms}] \,.\, \exp[(\nu+\varepsilon)\,.\,e^{\varrho \cdot \psi(x)}]$$

$$= O\,[\exp\{(\nu+\varepsilon)\, e^{\varrho \cdot \psi(x)}\}].$$

Also

$$\sum_{M+1}^{\infty} \left[\frac{(\nu+\varepsilon)\, e\, \varrho \,.\, e^{\varrho \cdot \psi(x)}}{\lambda_n}\right]^{\lambda_n/\varrho} < \sum_{M+1}^{\infty} \left[\frac{\nu+\varepsilon}{\nu+2\varepsilon}\right]^{\lambda_n/\varrho} = 0\,(1).$$

Thus,

(4.4) $$T = \lim_{x\to\infty} \sup \frac{\log F(x)}{e^{\varrho \cdot \psi(x)}} \leq \nu$$

Again suppose $0 < \nu \leq \infty$, we have, for an infinity of n

(4.5) $$a_n > \left[\frac{(\nu - \varepsilon)\, e\, \varrho}{\lambda_n}\right]^{\lambda_n/\varrho}, \qquad (0 < \varepsilon < \nu).$$

If we take $\exp[\varrho \,, \psi(x)] = \lambda_n/(\nu - \varepsilon)\, \varrho$, for these values of λ_n we have,

$$F(x) > a_n e^{\lambda_n \cdot \psi(x)} > \left[\frac{(\nu - \varepsilon)\, e \,.\, \varrho \,.\, e^{\varrho \cdot \psi(x)}}{\lambda_n}\right]^{\lambda_n/\varrho}$$

for sufficiently large values of $\psi(x)$.
Thus,

(4.6) $$T \geq \nu$$

Hence the result in (4.2) follows from (4.4) and (4.6).

Applications

5. Here we give some applications of the results derived in the previous sections to the entire functions.

First we consider the case of TAYLOR Series. Let $f(z) = \sum_{0}^{\infty} a_n z^n$, $z = x + iy$, be an entire function of order ϱ and lower order λ; «T» and «t» denote the type and lower type of the function $f(z)$; $M(r)$ denotes its maximum modulus and $\mu(r)$ is the maximum term of the rank $\nu(r)$ for $|z| = r$

The fact [1, pp. 253] that $\Sigma\, |a_n z^n|$ does not differ very much from its greatest term and that $|f(z)|$ lies betweeu the two, makes it possible to study the growth of the function $f(z)$ by investigating the growth of the function

$$\Sigma\, |a_n|\, |z|^n \qquad i.e. \qquad \Sigma\, |a_n|\, r^n \qquad (|z| = r)$$

of real variable r.

If, $F(r)$ is a function as defined in (1.1) with $\psi(r) = \log r$ and $\lambda_n = n$, then the series in (1.1) becomes a TAYLOR Series viz,

$$F(r) = \Sigma\, a_n r^n.$$

It is known [2, pp 31] that.

(5.1) $$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(t)}{t}.$$

REFERENCE

[1] E. C. TITCHMARSH : The Theory of functions, 2nd edition, OXFORD (1958)

[2] G. VALIRON : Lectures on the general theory of integral functions, CHELSEA PUB. Co., (1953)

[3] R. P. BOAS, JR. : Entire Functions, ACADEMIC PRESS INC, New York (1954)

[4] C. Y. YU : *Sur les droites de Borel de certaines fonctions entières*, Ann Sci. École Norm. Sup., 68 (1951)

[5] J. F. RITT. : *On certain points in the theory of Dirichlet series*, Amer. J. Math., 50 (1928)

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ÖZET

Bu araştırmada, bir reel değişkenin sınırsız olarak artan bir fonksiyonunun, diğer bir fonksiyona nazaran artışı incelenmiştir. Bu maksatla mertebe ve tip kavramları kullanılmıştır. Tam fonksiyonlar teorisi, tarifleri için TAYLOR veya DIRICHLET serisi kullanıldığına göre, birbirinden ayrı muhtelif cepheleri varmış gibi yürütülmüş ve bu alanda araştırmada bulunan kimseler bu ayırmaya riayet etmiştir. Bu travayda ise teorinin muhtelif veçhelerini birleştirmek için bir teşebbüse girişilmiştir: 5. ci kısım da bu yönde elde edilen sonuçları iktiva etmektedir.