AUTOMORPHISM GROUP OF THE UNIT DISK $|z| < 1$

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A simple proof of the well-known theorem that the automorphism group of the unit disk in the complex plane consists of the linear transformations

$$e^{i\vartheta}\frac{z+z_0}{1+z\bar{z}_0}$$

is given.

1. In this short note we submit a concise and apparently new proof of the following well-known theorem;

The automorphism group of the unit disk $|z| < 1$ *in the complex plane* C *consists of the linear transformations*

$$w = f(z) = e^{i\vartheta}\frac{z+z_0}{1+z\bar{z}_0} \tag{1}$$

where $0 \leq \vartheta \leq 2\pi$ *and* z_0 *is any complex number such that* $|z_0| < 1$.

2. *Proof;*

I. Let f be any automorphism of the disk $|z| < 1$ and $f(0) = \alpha$: then $|\alpha| < 1$.

Consider the function

$$g(z) = \frac{f(z) - \alpha}{1 - \bar{\alpha} f(z)} \quad \text{for } |z| < 1.$$

a) $|g(z)| < 1$ *whenever* $|z| < 1$.

Indeed, if $|a| < 1$ and $|b| < 1$, then

$$\left| \frac{a-b}{1-a\bar{b}} \right| < 1,$$

and since $|\alpha| < 1$ and $|f(z)| < 1$ for $|z| < 1$, the statement *a)* is proved.

b) $g(z)$ *is analytic in* $|z| < 1$.

Since $|\alpha| < 1$, we have $|\bar{\alpha}| < 1$. Futhermore, if $|z| < 1$, $|f(z)| < 1$, so $|\bar{\alpha} f(z)| < 1$ whenever $|z| < 1$. Consequently, $1 - \bar{\alpha} f(z) \neq 0$ in $|z| < 1$, and this proves statement *b)*.

c) $g(z)$ *is one-one in* $|z| < 1$.

Suppose

$$g(z_1) = g(z_2), \text{ where } |z_1| < 1, |z_2| < 1.$$

Then

$$\frac{f(z_1) - \alpha}{1 - \bar{\alpha} f(z_1)} = \frac{f(z_2) - \alpha}{1 - \bar{\alpha} f(z_2)}$$

so

$$(1 - |\alpha|^2)\,[f(z_1) - f(z_2)] = 0.$$

Since $|\alpha| < 1$, this implies

$$f(z_1) = f(z_2)$$

but as f is one-one,

$$z_1 = z_2,$$

and thus statement *c*) is proved.

d) *$g(z)$ maps the disk $|z| < 1$ onto the disk $|z| < 1$.*

Consider β such that $|\beta| < 1$. Then we will have $g(z) = \beta$ if

$$\frac{f(z) - \alpha}{1 - \bar{\alpha} f(z)} = \beta$$

i. e. if

$$f(z) = \frac{\alpha + \beta}{1 + \bar{\alpha}\beta}\,.$$

However, since $|\alpha| < 1$ and $|\beta| < 1$, we have $\left|\dfrac{\alpha + \beta}{1 + \bar{\alpha}\beta}\right| < 1$, so, since f maps the disk $|z| < 1$ onto the disk $|z| < 1$, there exists a z_0 such that

$$|z_0| < 1 \text{ and } f(z_0) = \frac{\alpha + \beta}{1 + \bar{\alpha}\beta}.$$

Obviously for this z_0, $g(z_0) = \beta$, and this proves statement *d*).

The above results show that $g(z)$ is an automorphism of the disk $|z| < 1$, such that $g(0) = 0$. Hence g must be a rotation, (see [1]), so

$$g(z) = \lambda z, \text{ for } |z| < 1 \text{ where } |\lambda| = 1,$$

i. e.

$$\frac{f(z) - \alpha}{1 - \bar{\alpha} f(z)} = \lambda z$$

or

$$f(z) = \frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z}\,.$$

II. Conversely, if α and λ are any complex numbers such that $|\alpha| < 1$ and $|\lambda| = 1$, then

$$f(z) = \frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z}$$

is an automorphism of the disk $|z| < 1$.

Obviously, f is one-one and analytic in $|z| < 1$. Futhermore, if $|z| < 1$, then $|f(z)| < 1$, because

$$|f(z)| = \left| \frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z} \right| < 1.$$

Finally f maps the disk $|z| < 1$ onto the disk $|z| < 1$: to prove this last statement consider β such that $|\beta| < 1$. Then $f(z) = \beta$ if

$$\frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z} = \beta$$

i. e. if

$$z = \frac{\beta - \alpha}{\lambda (1 - \bar{\alpha} \beta)}$$

and, since $|\alpha| < 1$, $|\beta| < 1$ and $|\lambda| = 1$,

$$\left| \frac{\beta - \alpha}{\lambda (1 - \bar{\alpha} \beta)} \right| = \left| \frac{\beta - \alpha}{1 - \bar{\alpha} \beta} \right| < 1.$$

Hence, if

$$z_0 = \frac{\beta - \alpha}{\lambda (1 - \bar{\alpha} \beta)},$$

then $|z_0| < 1$ and $f(z_0) = \beta$, as was to be shown.

f is therefore an automorphism of the disk $|z| < 1$.

III. We have thus proved that the automorphism group of the disk $|z| < 1$, consists of the linear transformations

$$w = f(z) = \frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z}$$

where α and λ are any complex numbers such that $|\alpha| < 1$ and $|\lambda| = 1$.

Since α is arbitrary, provided it satisfies the condition $|\alpha| < 1$, we may rewrite it as $\alpha = \lambda z_0$, where z_0 is arbitrary but satisfies the condition $|z_0| < 1$. Then

$$w = f(z) = \frac{\lambda z + \alpha}{1 + \lambda \bar{\alpha} z} = \frac{\lambda z + \lambda z_0}{1 + \lambda \bar{\lambda} z \bar{z}_0} = \lambda \frac{z + z_0}{1 + z \bar{z}_0} = e^{i\vartheta} \frac{z + z_0}{1 + z \bar{z}_0}$$

where $0 \leq \vartheta \leq 2\pi$ and this completes the proof.

BIBLIOGRAPHY

[1] H. CARTAN : **Elementary Theory of Analytic Functions of One or Several Complex Variables.** Addison-Wesley Publishing Co., London (1963).

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ÖZET

Bu araştırmada, kompleks düzlemdeki birim daire içinin otomorfizmalar grubunun

$$e^{i\vartheta} \frac{z + z_0}{1 + z \bar{z}_0}$$

şeklindeki dönüşümler tarafından doğurulduğuna dâir çok bilinen teoremin yeni bir ispatı verilmektedir.