A NOTE ON ENTIRE FUNCTIONS

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A property of an entire function, stated in the following theorem, is proved.

With the usual notations for an entire function $f(z)$, wc prove the following theorem :

Theorem 1. *For an entire function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\lim_{r \to \infty} M_\delta (r,f) / M(r,f) = 0,$$

if

$$R_n = | a_{n-1} / a_n |$$

is a strictly increasing function of n, *and*

$$L(f) = \limsup_{n \to \infty} | a_n^2 / a_{n-1} a_{n+1} | = 1,$$

where

$$M_\delta (r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) |^\delta \, d\theta \right)^{1/\delta}, \ (0 < \delta < \infty).$$

Proof. Since R_n is a strictly increasing function of n, $f(z) = p(z) + A \varphi_1(z)$, where A is a constant, $p(z)$ is a poiynominal, and

$$\varphi_1(z) = \sum_{n=1}^{\infty} z^n e^{i\vartheta_n} / R_1 R_2 \ldots\ldots R_n .$$

So,

$$M(r,f) \sim M(r, \varphi_1),$$

and

(1.1) $$M_\delta (r,f) \sim M(r, \varphi_1).$$

Let

$$\varphi(z) = \sum_{n=1}^{\infty} z^n / R_1 R_2 \ldots R_n .$$

For

$$R_n \leq r < R_{n+1}, \quad n = \nu(r, \varphi),$$

and

$$\mu(r, \varphi) = r^n / R_1 R_2 \ldots R_n .$$

Also,

$$M_2(r, \varphi) = \left(\sum_{n=1}^{\infty} r^{2n} / (R_1 R_2 \ldots R_n)^2 \right)^{1/2}.$$

Hence,

$$M_2(r, \varphi) / M(r, \varphi) = [1 + \{ (r^2 / R_n^2) + (R_{n-1}^2 / r^2) \}$$
$$+ \{ (r^4 / R_{n+1}^2 R_{n+2}^2) + (R_{n-1}^2 R_{n-2}^2 / r^4) \} + \cdots]^{1/2} /$$
$$[1 + (r/R_{n+1}) + (R_{n-1}/r) + \{ (r^2 / R_{n+1} R_{n+2}) + (R_{n-1} R_{n-2} / r^2) \} + \cdots],$$

for

$$R_n \leq r < R_{n+1} .$$

From the hypothesis we have $R_n \sim R_{n+1}$.

Hence,

(1.2) $$\lim_{r \to \infty} M_2(r, \varphi) / M(r, \varphi) = 0 .$$

Let

$$f_1(z) = \sum_{n=1}^{\infty} z^n / (R_1 R_2 \ldots R_n)^2 .$$

Then we have

$$[M_2(r, \varphi_1)]^2 = M_2(r^2, f_1),$$

and

$$[M(r, \varphi_1)]^2 \geq M(r^2, f_1).$$

Hence,

(1.3) $$M_2(r, \varphi_1) / M(r, \varphi_1) \leq [M_2(r^2, f_1) / M(r^2, f_1)]^{1/2} .$$

From (1.2), and (1.3) we have

(1.4) $$\lim_{r \to \infty} M_2(r, \varphi_1) / M(r, \varphi_1) = 0 .$$

Let

$$G_1(z) = \sum_{n=0}^{\infty} b_n z^n,$$

and

$$G_2(z) = \sum_{n=0}^{\infty} C_n z^n$$

be two entire functions, and let $G(z) = G_1(z) G_2(z)$.

Then

$$G(z) = \sum_{n=0}^{\infty} d_n z^n,$$

where

$$d_n = b_0 C_n + b_1 C_{n-1} + b_2 C_{n-2} + \cdots\cdots + b_n C_0.$$

So

$$\mu(r, G) \leq |b_0| \; |C_n| \; r^n + |b_1| r | C_{n-1}| \; r^{n-1} + \cdots + |b_n| \; r^n \; |C_0|$$

$$= \sum_{\gamma=0}^{n} |b_\gamma| \; r^\gamma \; |C_{n-\gamma}| \; r^{n-\gamma}$$

$$\leq \left(\sum_{\gamma=0}^{n} |b_\gamma|^2 r^{2\gamma} \right)^{1/2} \left(\sum_{\gamma=0}^{n} |C_\gamma|^2 r^{2\gamma} \right)^{1/2}$$

$$< M_2(r, G_1). \; M_2(r, G_2).$$

i. e.

(1.5) $$\mu(r, G) < M_2(r, G_1) \; M_2(r, G_2)$$

Case 1. Suppose $\delta \geq 2$ is an even integer.

Let

$$\psi(z) = [\varphi_1(z)]^{\delta/2}$$

So

(1.6) $$M_2(r, \psi) = \left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi_1(r e^{i\vartheta})|^\delta \, d\vartheta \right)^{1/2} = [M_\delta(r, \varphi_1)]^{\delta/2}.$$

Now, using (1.5) and (1.6)

$$\mu(r, \psi) / M(r, \psi) = \mu(r, \psi) / [M(r, \varphi_1)]^{\delta/2}$$
$$< M_2(r, \varphi_1). \; M_2(r, \varphi_1^{(\delta/2)-1}) / [M(r, \varphi_1)]^{\delta/2}$$
$$= M_2(r, \varphi_1) [M_{\delta-2}(r, \varphi_1)]^{(\delta/2)-1} / [M(r, \varphi_1)]^{\delta/2}.$$

So

$$\mu(r, \psi) / M(r, \psi) < M_2(r, \varphi_1) / M(r, \varphi_1),$$

since

$$M_\delta(r, \varphi_1) \leq M(r, \varphi_1)$$

for δ satisfying

$$0 < \delta < \infty.$$

Hence from (1.4) we have,

$$\lim_{r \to \infty} \mu(r, \psi) / M(r, \psi) = 0 \tag{1.7}$$

Let $L(\psi) > 1$. Then from a known result of S. M. SHAH [1], we have

$$\limsup_{r \to \infty} \mu(r, \psi) / M(r, \psi) > 0.$$

This contradicts (1.7). So $L(\psi) = 1$.

Hence, applying (1.4) to $\psi(z)$, and using (1.6) we have

$$\lim_{r \to \infty} M_\delta(r, \varphi_1) / M(r, \varphi_1) = 0. \tag{1.8}$$

From (1.1), and (1.8) we then have

$$\lim_{r \to \infty} M_\delta(r, f) / M(r, f) = 0. \tag{1.9}$$

Case 2. Suppose $\delta < 2$.

The result follows easily from (1.9), since $M_\delta(r, f)$ is an increasing function of δ.

Case 3. Suppose $\delta > 2$ is not an even integer.

We select an even integer $\delta_1 > \delta$, so that (1.9) gives

$$\lim_{r \to \infty} M_\delta(r, f) / M(r, f) \leq \lim_{r \to \infty} M_{\delta_1}(r, f) / M(r, f) = 0.$$

Hence the theorem is proved completely.

REFERENCE

[1] S. M. SHAH : *The behaviour of entire functions and a conjecture of* P. ERDÖS, Amer. Math. Monthly, 68, (1961).

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(Manuscript received December 14, 1965)

ÖZET

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

tam fonksiyonu için, δ sayısı $0 < \delta < \infty$ şartını sağlamak üzere

$$M_\delta (r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} | f(r e^{i\vartheta}) |^\delta \, d\vartheta \right)^{1/\delta}$$

vaz ediliyor.

$$R_n = | a_{n-1} / a_n |$$

n'nin kesin olarak artan bir fonksiyonu ve

$$L(f) = \limsup_{n\to\infty} | a_n^2 / a_{n-1} a_{n+1} | = 1$$

ise,

$$\lim_{r\to\infty} M_\delta (r, f) / M(r, f) = 0$$

olduğu ispat edilmektedir.