ON H_2 SUMMABILITY OF LEGENDRE SERIES

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Foà (1943) studied the summability problem for the Legendre series and obtained a result which corresponds to that of Hardy and Litlewood (1913) in the case of Fourier series. The object of this paper is to obtain, for the strong summability of the Legendre series, a new result in a different line, which corresponds to that obtained by Wang (1944) for Fourier series.

1. Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. The series $\Sigma\, u_n$ is said to be strongly summable by Cesàro means, with index 2, or summable $[C, 2]$, or summable H_2, to a sum S, if

$$\sum_{\nu=0}^{n} | S_\nu - S |^2 = O(n). \tag{1.1}$$

Let $f(x)$ be a function integrable (L) over the range $[-1, 1]$. The Legendre series associated with this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x), \tag{1.2}$$

where

$$a_n = \left(n + \tfrac{1}{2} \right) \int_{-1}^{+1} f(x)\; P_n(x)\; dx \tag{1.3}$$

and $P_n(x)$ is the n-th Legendre polynomial.

We use the following notations:

$$\varphi(t) \equiv f\{\cos(r - t)\} - f(\cos r),$$

$$\Phi(t) \equiv \int_0^t | \varphi(v) |\, dv$$

and

$$\varkappa(t) = \varphi(t) \sin^{1/2}(r - t).$$

2. FOÀ (1943) studied the summability problem for the LEGENDRE series and obtained a result which corresponds to that of HARDY and LITTLEWOOD (1913) in the case of FOURIER series. The object of this paper is to obtain, for the strong summability of the LEGENDRE series, a new result in a different line, which corresponds to that obtained by WANG (1944) for FOURIER series.

We prove the following theorem :

Theorem. *If, for some* $\alpha > \frac{1}{2}$,

$$\int_0^t |f(x \pm u) - f(x)| \, du = O\left[\frac{t}{\{\log(1/t)\}^{\alpha}}\right], \tag{2.1}$$

as $t \to 0$, *then*

$$\sum_{\nu=0}^{n} | s_\nu(x) - f(x) |^2 = O(n), \quad (-1 + \varepsilon \leq x \leq 1 - \varepsilon, \ \varepsilon > 0)$$

where $s_n(x)$ *denotes the n-th partial sum of the Legendre series.*

3. We require the following lemmas for the proof of our theorem.

Lemma 1. [1959, p. 179].

$$\sum_{\nu=0}^{n} (2\nu + 1) P_\nu(x) P_\nu(y) = (n+1) \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{y - x}. \tag{3.1}$$

Lemma 2. [1959, p. 208].

$$P_n(\cos r) = \sqrt{\frac{2}{\pi n \sin r}} \cos\left[(n + \tfrac{1}{2}) r - \frac{\pi}{4}\right] + O(n^{-3/2}). \tag{3.2}$$

Lemma 3. Under the condition of the theorem, we have

$$\int_0^t | f\{\cos(r - v)\} - f(\cos r) | \, dv = O\left[\frac{t}{\{\log \frac{1}{2}\}^{\alpha}}\right], \quad \text{as } t \to 0,$$

where $x = \cos r$, $x + u = \cos r'$ and $r - r' = v$.

The proof of the lemma follows on the lines of FOÀ [1943, section c].

Lemma 4. If

$$\int_0^t | \varphi(u) | \, du = O(t), \text{ as } t \to 0,$$

then, for some small but fixed η,

$$\int_{1/n}^{\eta} \frac{|\varphi(t)|}{t}\, dt \int_{1/n}^{\eta} \frac{|\varphi(u)|}{u}\, du = O(n),$$

and

$$\int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2}\, dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u}\, du = O(n).$$

Proof. Since

$$\int_{1/n}^{\eta} \frac{|\varphi(u)|}{u}\, du = \left[\frac{\Phi(u)}{u}\right]_{1/n}^{\eta} + \int_{1/n}^{\eta} \frac{\Phi(u)}{u^2}\, du$$

$$= O(1) + \int_{1/n}^{\eta} O(1/u)\, du$$

$$= O(1) + O(\log n)$$

$$= O(\log n).$$

Therefore

$$\int_{1/n}^{\eta} \frac{|\varphi(t)|}{t}\, dt \int_{1/n}^{\eta} \frac{|\varphi(u)|}{u}\, du = O\{(\log n)^2\} = O(n).$$

Again

$$\int_{1/n}^{n} \frac{|\varphi(t)|}{t^2}\, dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u}\, du$$

$$= \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} \left\{\left[\frac{\Phi(u)}{u}\right]_{1/n}^{t} + \int_{1/n}^{t} \frac{\Phi(u)}{u^2}\, du\right\} dt$$

$$= \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} \left\{\frac{\Phi(t)}{t} + O(1) + O(\log nt)\right\} dt$$

$$= O\left\{\int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} \log nt\, dt\right\}$$

$$= O\left\{\left[\frac{\Phi(t)}{t^2} \log nt\right]_{1/n}^{\eta} - \int_{1/n}^{\eta} \frac{\Phi(t)}{t^3}\, dt + 2\int_{1/n}^{\eta} \frac{\Phi(t)}{t^3} \log nt\, dt\right\}$$

$$= O(n) + O\left(n \int_{1}^{n\eta} \frac{dv}{v^2}\right) + O\left(n \int_{1}^{n\eta} \frac{\log v}{v^2}\, dv\right)$$

$$= O(n).$$

4. Proof of the theorem. The ν - th partial sum of the series (1.2) is

$$s_\nu(x) = \sum_{k=0}^{\nu} a_k P_k(x)$$

$$= \frac{\nu+1}{2} \int_{-1}^{+1} f(x') \frac{P_\nu(x) P_{\nu+1}(x') - P_{\nu+1}(x) P_\nu(x')}{x' - x} dx'.$$

by Lemma 1.

Putting $f(x') = 1$, it can be easily seen that

$$1 = \frac{\nu+1}{2} \int_{-1}^{+1} \frac{P_\nu(x) P_{\nu+1}(x') - P_{\nu+1}(x) P_\nu(x')}{x' - x} dx'.$$

Therefore

$$s_\nu(x) - f(x) = \frac{\nu+1}{2} \int_{-1}^{+1} [f(x') - f(x)] \frac{P_\nu(x) P_{\nu+1}(x') - P_{\nu+1}(x) P_\nu(x')}{x' - x} dx'.$$

Let us take a positive number s less than unity and consider it as the sum of two positive numbers μ and ξ such that $\mu + \xi = s$.

Let d be another positive number such that $0 < d < \mu$; and let μ_x and μ_x' be two continuous functions of x within $(-1, 1)$ which lie within the limits

$$d \leq \mu_x \leq \mu; \; d \leq \mu_x' \leq \mu.$$

Then for

$$-1 + s \leq x \leq 1 - s,$$

we have

$$s_\nu(x) - f(x) = A_\nu(x) + B_\nu(x) + C_\nu(x),$$

with

$$A_\nu(x) = \frac{\nu+1}{2} \int_{-1}^{x-\mu x} \varphi(x, x') \, g_\nu(x, x') \, dx',$$

$$B_\nu(x) = \frac{\nu+1}{2} \int_{x-\mu x}^{x+\mu x'} \varphi(x, x') \, g_\nu(x, x') \, dx',$$

$$C_\nu(x) = \frac{\nu+1}{2} \int_{x+\mu x'}^{+1} \varphi(x, x') \, g_\nu(x, x') \, dx',$$

$$\varphi(x, x') = f(x') - f(x),$$

and

$$g_\nu(x, x') = \frac{P_\nu(x)\, P_{\nu+1}(x') - P_{\nu+1}(x)\, P_\nu(x')}{x - x'}.$$

HOBSON [1931, p.320-322] has shown that uniformly for

$$-1 + s \leq x \leq 1 - s,$$

$$\lim_{\nu\to\infty} A_\nu(x) + \lim_{\nu\to\infty} C_\nu(x) = 0.$$

Now let us suppose $x = \cos r$, $x' = \cos r'$; $0 < r < \pi$, $0 < r' < \pi$, $1 - \xi = \cos \varrho$, $1 - (\mu + \xi) = 1 - s = \cos(\varrho + \tau)$, $0 < \varrho < \pi/2$, Thus, if η denotes the minimum of [arc cos u — arc cos $(u + \mu)$] for u in $(-1, 1 - \mu)$, we have on the lines of SANSONE [1959, p. 226]

$$s_\nu(x) - f(x) = B_\nu(\cos r) = \frac{\nu + 1}{2} \int_{r-\eta}^{r+\eta} \varphi(r, r')\, g_\nu(r, r') \sin r'\, dr'$$

in which

$$\varphi(r, r') = f(\cos r') - f(\cos r),$$

$$g_\nu(r, r') = \frac{P_\nu(\cos r)\, P_{\nu+1}(\cos r') - P_{\nu+1}(\cos r)\, P_\nu(\cos r')}{\cos r' - \cos r}$$

$$\varrho + \tau \leq r \leq \pi - (\varrho + \tau), \quad 0 < \eta \leq \tau.$$

Using Lemma 2 and working on the lines of SZEGÖ [1939, p. 252, for $\alpha = \beta = 0$], we get after some simplifications

$$S_\nu(x) - f(x) = \frac{1}{2\pi \sin^{1/2} r} \int_{r-\eta}^{r+\eta} \varphi(r, r') \sin^{1/2} r' \left[\frac{\sin\{(\nu+1)(r-r')\}}{\sin \frac{1}{2}(r-r')} \right.$$
$$\left. - \frac{\cos\{(\nu+1)(r+r')\}}{\sin \frac{1}{2}(r+r')} + O\left(\frac{1}{\nu^2}\right) \right] dr'.$$

Now

$$\int_{r-\eta}^{r+\eta} \frac{\varphi(r, r') \sin^{1/2} r'}{\sin \frac{1}{2}(r+r')} \cos\{(\nu+1)(r+r')\}\, dr' = o(1)$$

as $\nu \to \infty$, by virtue of RIEMANN-LEBESGUE theorem.

And clearly as $\nu \to \infty$

$$O\left(\frac{1}{\nu^2}\right) \int_{r-\eta}^{r+\eta} |\varphi(r, r') \sin^{1/2} r'|\, dr' = o(1).$$

Hence putting $r-r'=t$ and denoting $\pi^{-1}\sin^{-1/2} r$ by H, we get

$$s_\nu(x)-f(x)=H\int_0^\eta \frac{\varphi(t)\sin(\nu+1)t\sin^{1/2}(r-t)}{\sin\frac{1}{2}t}\,dt.$$

Since

$$\left|\int_0^{1/n}\varphi(t)\frac{\sin(\nu+1)t}{\sin\frac{1}{2}t}\sin^{1/2}(r-t)\,dt\right|$$
$$\leqq\int_0^{1/n}2(\nu+1|\varphi(t)|\,dt=o\left[\frac{\nu}{n\{\log n\}^\alpha}\right]=o(1),$$

by hypothesis of the theorem and lemma 3.

Therefore

$$s_\nu(x)-f(x)=H\int_{1/n}^\eta \frac{\varphi(t)\sin(\nu+1)t\sin^{1/2}(r-t)}{\sin(t/2)}\,dt+O(1)$$
$$=2H\int_{1/n}^\eta\frac{\varphi(t)}{t}\sin(\nu+1)t\sin^{1/2}(r-t)\,dt+O(1)$$
$$=2H\int_{1/n}^\eta\frac{\varkappa(t)}{t}\sin(\nu+1)t\,dt+O(1)$$

where $\varkappa(t)=\sin(r-t)\,\varphi(t)$ and therefore $|\varkappa(t)|\leqq|\varphi(t)|$.

Hence

$$\sum_{\nu=1}^{n}(s_\nu(x)-f(x))^2$$
$$=4H^2\int_{1/n}^\eta\int_{1/n}^\eta\frac{\varkappa(t)}{t}\frac{\varkappa(u)}{u}\left\{\sum_{\nu=1}^{n}\sin(\nu+1)t\sin(\nu+1)u\right\}dt\,du+O(n)$$

(4.1) $$=2H^2\int_{1/n}^\eta\int_{1/n}^\eta\frac{\varkappa(t)}{t}\frac{\varkappa(u)}{u}\left\{\sum_{\nu=1}^{n}[\cos\{\nu+1)(u+t)\}-\cos\{(\nu+1)(u-t)\}]\,dt\,du+O(u)\right.$$

$$=2H^2\int_{1/n}^\eta\int_{1/n}^\eta\frac{\varkappa(t)}{t}\frac{\varkappa(u)}{u}\left[\frac{\sin(n+3/2)(u+t)}{2\sin\frac{1}{2}(u+t)}-\cos(u+t)\right]dt\,du$$
$$-2H^2\int_{1/n}^\eta\int_{1/n}^\eta\frac{\varkappa(t)}{t}\frac{\varkappa(u)}{u}\left[\frac{\sin(n+3/2)(u-t)}{2\sin\frac{1}{2}(n-t)}-\cos(u-t)\right]dt\,du+O(n)$$

$$= 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{\eta} \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u+t)\}}{u+t} du$$

$$- 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{\eta} \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u-t)\}}{u-t} du + O(n)$$

$$= J_1 - J_2 + O(n),$$

say, by application of the first part of the Lemma 4.

Now

$$J_2 = 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{\eta} \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u-t)\}}{u-t} du$$

$$= 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \left[\int_{1/n}^{t} + \int_{t}^{\eta} \right] \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u-t)\}}{u-t} du$$

$$= 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{t} \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u-t)\}}{u-t} du$$

$$+ 2H^2 \int_{1/n}^{\eta} \frac{\varkappa(u)}{u} du \int_{1/n}^{u} \frac{\varkappa(t)}{t} \frac{\sin\{(n+1)(u-t)\}}{u-t} dt$$

$$= 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{t} \frac{\varkappa(u)}{u} \frac{\sin\{(n+1)(u-t)\}}{u-t} du$$

$$= 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{t} \frac{\varkappa(u)}{t} \left[\frac{1}{u-t} - \frac{1}{u} \right] \frac{\sin\{(n+1)(u-t)\}}{u-t} du$$

$$= 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t^2} dt \int_{1/n}^{t} \frac{\varkappa(u)}{u-t} \sin\{(n+1)(u-t)\} du$$

$$+ O\left\{ \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u} du \right\}$$

$$= 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t^2} dt \int_{1/n}^{t} \varkappa(u) \frac{\sin\{(n+1)(u-t)\}}{u-t} du + O(n),$$

by virtue of the second part of Lemma 4.

Also by Lemma 3 and the hypothesis of the theorem, we get

$$\int_{1/n}^{t} \varkappa(u) \frac{\sin\{(n+1)(u-t)\}}{u-t} du = O\left(n \int_{0}^{t} |\varphi(n)| du\right) = O\left(\frac{nt}{\{\log \frac{1}{t}\}^{\alpha}}\right).$$

Hence

$$J_2 = O\left[n \int_{1/n}^{\eta} \frac{|\Phi(t)|}{t\{\log \frac{1}{t}\}^{\alpha}} dt\right] + O(n)$$

$$= O(n)\left[\frac{\Phi(t)}{t\{\log \frac{1}{t}\}^{\alpha}}\right]_{1/n}^{\eta} + O(n)\left[\int_{1/n}^{\eta} \frac{|\Phi(t)|}{t^2\{\log \frac{1}{t}\}^{\alpha}} dt\right]$$

(4.2) $$+ O(n)\left[\int_{1/n}^{\eta} \frac{\Phi(t)}{t^2\{\log \frac{1}{t}\}^{\alpha+1}} dt\right]$$

$$= O(n) + O(n)\left[\int_{1/n}^{\eta} \frac{dt}{t(\log \frac{1}{t})^{2\alpha}}\right] + O(n)\left[\int_{1/n}^{\eta} \frac{dt}{t(\log \frac{1}{t})^{2\alpha+1}}\right]$$

$$= O(n) + O(n)\left[\int_{1/\eta}^{n} \frac{dv}{t(\log v)^{2\alpha}}\right] + O(n)\left[\int_{1/\eta}^{n} \frac{dv}{v(\log v)^{2\alpha+1}}\right]$$

$$= O(n).$$

Similarly

$$J_1 = 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t} dt \int_{1/n}^{t} \frac{\varkappa(u)}{t}\left[\frac{1}{u} - \frac{1}{u+t}\right] \sin\{(n+1)(u+t)\} du$$

$$= 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t^2} dt \int_{1/n}^{t} \frac{\varkappa(u)}{u} \sin\{(n+1)(n+t)\} du$$

$$- 4H^2 \int_{1/n}^{\eta} \frac{\varkappa(t)}{t^2} dt \int_{1/n}^{t} \frac{\varkappa(u)}{u+t} \sin\{(n+1)(u+t)\} du.$$

Therefore

$$|J_1| \leq 4H^2 \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u} du + 4H^2 \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2} dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u} du.$$

Hence

$$J_1 = O\left\{ \int_{1/n}^{\eta} \frac{|\varphi(t)|}{t^2}\, dt \int_{1/n}^{t} \frac{|\varphi(u)|}{u}\, du \right\} \tag{4.3}$$

$$= O(n),$$

by virtue of the second part of Lemma 4.

The required result now follows from (4.1), (4.2) and (4.3).

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(Manuscript received March 28, 1966)

ÖZET

LEGENDRE serisinin yakınsaklık problemi 1943 senesinde FOÀ tarafından incelenmiş ve daha önce HARDY ve LITTLEWOOD tarafından 1913 te FOURIER serileri için elde edilenlere benzer sonuçlara varılmıştır.

Bu yazıda ise bir serinin kısmî toplamlarının CESÀRO ortalaması ile elde edilen kesin yakınsaklık kavramı LEGENDRE serisine uygulanarak, WANG tarafından 1944 te FOURIER serilerinin kesin yakınsaklığı hususunda elde edilen teoremleri andıran sonuçlar bulunmuştur.