CONFORMAL IDENTITIES

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Conformal transformation has been studied by M. S. Knebelman [1], H. Hombu [2], H. Rund [3], and R. S. Sinha [4] and others (1). We shall discuss certain properties of the conformally changed curvature tensors and some identities satisfied by them.

1. Conformal transformation. Let the two distinct metric functions $F(x, \dot{x})$ and $\bar{F}(x, \dot{x})$ (2) be defined over a Finsler space F_n, so as to satisfy the following conditions [3]

(*i*) Both functions are positive provided all $\dot{x}^i \neq 0$ simultaneously.

(*ii*) Both functions are positively homogeneous of the first degree in $\dot{x}^i$.

(*iii*) The form

$$g_{ij}(x^i, \dot{x}^i)\, \xi^i \xi^j > 0 \quad \text{for} \quad \xi^i \neq 0$$

with any given argument $\dot{x}^i$ (3) where

$$g_{ij}(x^i, x'^i) \equiv \frac{1}{2} \frac{\partial^2 F^2(x, x')}{\partial x'^i\, \partial x'^j} .$$

The two metrics resulting from these functions are called conformal, if there exits a factor of proportionality $\psi(x, \dot{x})$, between the two metric tensors *i.e.*

$$\bar{g}_{ij}(x, \dot{x}) = \psi(x, \dot{x})\, g_{ij}(x, \dot{x}). \tag{1.1}$$

It has been shown that the factor of proportionality is at most a point function and we write (1.1) as

(1.2) *a*
$$\bar{g}_{ij} = e^{2\sigma} g_{ij},$$

where

(1.2) *b*
$$\sigma = \sigma(x) = \tfrac{1}{2} \log \psi.$$

(1) Numbers in brackets always refer to the references at the end of this paper.

(2) We denote the corresponding conformally transformed function by putting a horizontal bar on the same function.

(3) Repeated indices always imply summation.

Also we have

$$\bar{g}^{ij}(x, \dot{x}) = \ddot{e}^{2\sigma} g^{ij}(x, \dot{x}). \tag{1.3}$$

The BERWALD connection coefficients $\bar{G}^{i}_{jk}(x, \dot{x})$ are

$$\begin{aligned} \bar{G}^{h}_{ij} &=: G^{h}_{ij} + (\sigma_i \delta^h_j + \sigma_j \delta^h_i - g^{hk} g_{ij} \sigma_k) \\ &\quad - (\dot{\partial}_j g^{hk} g_{i\gamma} \dot{x}^\gamma + g_{i\gamma} \dot{x}^\gamma \dot{\partial}_i g^{hk} + \tfrac{1}{2} F^2 \dot{\partial}_i \dot{\partial}_j g^{hk}) \sigma_k, \end{aligned} \tag{1.4 a}$$

where

$$\sigma_i = \frac{\partial \sigma}{\partial x^i}. \tag{1.4 b}$$

Let us put

$$B^{hk} \overset{\text{def}}{=} \tfrac{1}{2} F^2 g^{hk} - \dot{x}^h \dot{x}^k. \tag{1.5}$$

It follows from (1.4) a and (1.5) that

$$\bar{G}^{h}_{ij} = G^{h}_{ij} - \dot{\partial}_i \dot{\partial}_j B^{hk} \sigma_k. \tag{1.6}$$

Differentiating (1.1) with respect to $\dot{x}^k$, we get

$$\bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \tag{1.7}$$

where

$$\bar{G}_{ijk} \overset{\text{def}}{=} \frac{\partial \bar{g}_{ij}}{\partial \dot{x}^k}. \tag{1.8}$$

We have [3] (1)

$$\begin{aligned} \bar{\Gamma}^{*h}_{ij}(x, \dot{x}) &= \Gamma^{*h}_{ij}(x, \dot{x}) + 2\, \sigma_{(i} \delta^h_{j)} - g^{hk} g_{ij} \sigma_k \\ &\quad + \left\{ 2\, \dot{\partial}_{(i} B^{lk} C^h_{j)l} - g^{hm} C_{ijl} \dot{\partial}_m B^{lk} \right\} \sigma_k, \end{aligned} \tag{1.9}$$

which can be written as [4]

$$\bar{\Gamma}^{*h}_{ij} = \Gamma^{*h}_{ij} + U^h_{ij}, \tag{1.10}$$

where

$$\left\{ \begin{aligned} U^h_{ij}(x, \dot{x}) &\overset{\text{def}}{=} 2\, \sigma_{(i} \delta^h_{j)} - g^{hk} g_{ij} \sigma_k \\ &\quad + \left\{ 2\, \dot{\partial}_{(i} B^{lk} C^h_{j)l} - g^{hm} C_{ijl} \dot{\partial}_m B^{lk} \right\} \sigma_k, \end{aligned} \right. \tag{1.11}$$

and we have [4] (2)

(1) We have $2 H_{(ij)} = H_{ij} + H_{ji}$ and $2 H_{[ij]} = H_{ij} - H_{ji}$.

(2) Greek indices always run from 1 to n.

(1.12) $$\bar{A}^i_{k\gamma} \overset{\text{def}}{=} \bar{F}\ \bar{C}^2_{k\gamma} = e^\sigma A^i_{h\gamma}\,.$$

2. Identities satisfied by the conformally changed curvature tensors

Considering the conformally changed curvature tensors

$$\bar{P}^i_{jhk}(x, \dot{x}),\quad \bar{K}^i_{jhk}(x, \dot{x})$$

and

$$\bar{R}^i_{jhk}(x, \dot{x})$$

given by [1]

(2.1)
$$\begin{cases} \bar{P}^i_{jkh}(x, \dot{x}) = e^\sigma\,[\,P^i_{jkh} + \sigma_j A^i_{kh} - 3\,\sigma_m\, g^{im} A_{jkh} \\ \quad - \sigma_\gamma\, l^\gamma S^i_{jkh} + \sigma_n\,(\dot{\partial}_\gamma A^i_{kh}\,\dot{\partial}_j B^{\gamma n} - g^{im}\,\dot{\partial}_\gamma A_{jkh} \\ \quad - \dot{\partial}_m B^{\gamma n}) - F\, g^{sn}\,\sigma_n\,(A^i_{mk}\,\dot{\partial}_s A^m_{jh} - A^m_{jk}\,\dot{\partial}_s A^i_{mh}) \\ \quad + A^\gamma_{kh}\, U^i_{\gamma j} - 2\, A^i_\gamma\,(h\, U^\gamma_k)_j + A_{\gamma kh}\, g^{im}\, U^\gamma_{jm} \\ \quad + 2\, g^{im} A_{j\gamma}\,(h\, U^\gamma_k)_m - l^\gamma \left\{ 2\, A^i_m\,(k\, A^s_h)_j\, U^m_{s\gamma} \right. \\ \quad \left. + U^s_{h\gamma}\, S^i_{jsk} - A^i_{mk}\, A^m_{sh}\, U^s_{j\gamma} - A^m_{jk}\, A^s_{mh}\, U^i_{s\gamma} \right\}\,], \end{cases}$$

where

(2.2) $$S^i_{jkh} \overset{\text{def}}{=} A^i_{k\gamma}\, A^\gamma_{jh} - A^i_{\gamma h}\, A^\gamma_{jk}$$

and

(2.3) $$P_{jlkh} = g_{il}\, P^i_{jkh}\,;$$

(2.4)
$$\begin{aligned} \bar{K}^i_{jhk} = K^i_{jhk} + 2\, U^i_{j\,[h\,|\,k]} + 2\,\sigma_n \left\{ \dot{\partial}_\gamma\, \Gamma^{*i}_{j\,[h} \right. \\ \left. + \dot{\partial}_\gamma\, U^i_{j\,[h} \right\} \dot{\partial}_{k]}\, B^{\gamma n} + 2\, U^i_{m\,[k}\, U^m_{h]j}\,, \end{aligned}$$

where

(2.5) $$U_{jlh} \overset{\text{def}}{=} g_{il}\, U^i_{jh}$$

and

(2.6)
$$\begin{aligned} \bar{R}^i_{jhk} = R^i_{jhk} + 2\, U^i_{j\,[h\,|\,k]} + 2 \left\{ \dot{\partial}_\gamma\, \Gamma^{*i}_{j\,[h} + \dot{\partial}_\gamma\, U^i_{j\,[h} \right\} \\ + \dot{\partial}_{k]} B^{\gamma n}\,\sigma_n + 2\, U^i_{m\,[k}\, U^m_{h]j} + 2\, C^l_{jm}\,[\,\{ \dot{\partial}_{[k}\, \partial_{h]}\, B^{m\gamma} \\ - G^\gamma_{l\,[k}\, \dot{\partial}_{h]}\, B^{ml} + \dot{\partial}_l\, \dot{\partial}_{[h}\, B^{m\gamma}\, \dot{\partial}_{k]}\, G^l - \dot{\partial}_{[h}\, B^{l\gamma} \\ - G^m_{k]l} \}\,\sigma_\gamma - \sigma_{\gamma[\,/\,k}\, \dot{\partial}_{h]}\, B^{m\gamma} + \dot{\partial}_{[h}\, B^{ln}\, \dot{\partial}_{k]}\, \dot{\partial}_l\, B^{m\gamma}\,\sigma_n\,\sigma_\gamma\,], \end{aligned}$$

where

(2.7) $$\sigma_{\gamma/k} \overset{\text{def}}{=\!=} \frac{\partial^2\sigma}{\partial x^k\, \partial x^\gamma} - \sigma_l\; G^l_{\gamma k}$$

respectively, we have the following theorems.

Theorem 2.1 — *The following identities hold* (*):

$$\bar{P}^i_{[jkh]} = e^{\sigma}\; P^i_{[jkh]}$$

and

$$\bar{P}_{[j\,\underline{|l|}\,kh]} = e^{3\sigma}\; P_{[j\,\underline{|l|}\,kh]}\,.$$

Proof. With the help of the equams (1.12), (1.13) and (2.2), we get

(2.8) *a* $$S^i_{[jkh]} = 0\,,$$

(2.8) *b* $$A^i_{m\,[k}\;\; A^m_{h\,\underline{|s|}}\;\; U^s_{j]\gamma} = 0\,,$$

and

(2.8) *c* $$U^s_{\gamma\,[h}\;\; S^i_{j\,\underline{|s|}\,k]} = 0\,.$$

Now using the relations (2.8) *a*, (2.8) *b* and (2.8) c in (2.1), we obtain

(2.9) $$\bar{P}^i_{[jkh]} = e^{\sigma}\; P^i_{[jkh]}\,.$$

Again in view of (2.3), we have

(2.10) $$\bar{P}_{[j\,\underline{|l|}\,kh]} = \bar{g}_{il}\; \bar{P}^i_{[jkh]}\,,$$

which yields from (2.9) that

(2.11) $$\bar{P}_{[j\,\underline{|l|}\,kh]} = e^{3\sigma}\; P_{[j\,\underline{|l|}\,kh]}\,.$$

Hence we have the Theorem.

Theorem 2.2 — $\bar{K}^i_{[jhk]}$ *are invariants under the conformal change* (1.1) *and*

$$\bar{K}_{[j\,\underline{|l|}\,hk]} = e^{2\sigma}\; K_{[j\,\underline{|l|}\,hk]}\,,$$

where

$$\bar{K}_{jlhk} = \bar{g}_{il}\; \bar{K}^i_{jhk}\,.$$

(*) $T_{[ijk]} = \frac{1}{\lfloor 3}\left\{ (T_{ijk} + T_{jki} + T_{kij}) - (T_{jik} + T_{ikj} + T_{kji}) \right\}$ and the index inside $\square$ is excluded from the skew-symmetric part.

Proof. With the help of (1.12), the equation (2.4) yields

$$\bar{K}^{i}_{[jhk]} = K^{i}_{[jhk]} \,. \tag{2.12}$$

Consequently, we have

$$\bar{K}_{[j\underline{|l|}hk]} = \bar{g}_{il}\, \bar{K}^{i}_{[jhk]}. \tag{2.13}$$

In view of the equations (1.2) a and (2.12), it reduces the form

$$\bar{K}_{[j\underline{|l|}hk]} = e^{2\sigma}\, K_{[j\underline{|l|}hk]} \,. \tag{2.14}$$

which proves the Theorem.

Theorem 2.3 — *The following identity holds:*

$$\begin{aligned}
\bar{R}_{jlhk} + \bar{R}_{jkhl} &= e^{2\sigma}\,(R_{jlhk} + R_{jkhl}) \\
&+ 2\,e^{2\sigma}\,\{ U_{j\,(l\underline{|h|}|k)} - U_{j\,(l\,k)\,|\,h} \} + 2\,e^{2\sigma}\,[g_{i\,(l} \\
&+ \{ \dot{\partial}_{k)}\, B^{\gamma n}\, \dot{\partial}_{\gamma}\, \Gamma^{*i}_{jh} - \dot{\partial}_{\underline{|\gamma|}}\, \Gamma^{*i}_{k)\,j}\, \dot{\partial}_{h}\, B^{\gamma n} + \dot{\partial}_{k)}\, B^{\gamma n} \\
&+ \dot{\partial}_{\gamma}\, U^{i}_{jh} - \dot{\partial}_{\underline{|\gamma|}}\, U^{i}_{k)\,j}\, \dot{\partial}_{h}\, B^{\gamma n} \}\, \sigma_{n} + U_{m\,(l\,k)}\, U^{m}_{hj} \\
&- U_{m\,(l\,\underline{|h|}}\, U^{m}_{k)\,j} + C_{j\,(l\,\underline{|m|}}\, [\, \{ \dot{\partial}_{k)}\, \dot{\partial}_{h}\, B^{m\gamma} - \dot{\partial}_{k)}\, \dot{\partial}_{h}\, B^{m\gamma} \\
&- G^{\gamma}_{k)\,p}\, \dot{\partial}_{h}\, B^{mp} + \dot{\partial}_{k)}\, B^{mp}\, G^{\gamma}_{ph} + \dot{\partial}_{k)}\, G^{p}\, \dot{\partial}_{p}\, \dot{\partial}_{h}\, B^{m\gamma} - \dot{\partial}_{k)}\, \dot{\partial}_{p}\, B^{m\gamma} \\
&\quad \dot{\partial}_{h}\, G^{p} - G^{m}_{k)\,p}\, \dot{\partial}_{h}\, B^{p\gamma} + \dot{\partial}_{k)}\, B^{p\gamma}\, G^{m}_{hp} \}\, \sigma_{\gamma} - \sigma_{\underline{|\gamma|}\,/\,k)}\, \dot{\partial}_{h}\, B^{m\gamma} \\
&+ \dot{\partial}_{k)}\, B^{m\gamma}\, \sigma_{\gamma/h} + \dot{\partial}_{k)}\, \dot{\partial}_{p}\, B^{m\gamma}\, \dot{\partial}_{h}\, B^{pn}\, \sigma_{n}\, \sigma_{\gamma} - \dot{\partial}_{k)}\, B^{pn} \\
&+ \dot{\partial}_{h}\, \dot{\partial}_{p}\, B^{m\gamma}\, \sigma_{n}\, \sigma_{\gamma} \})\,].
\end{aligned}$$

Proof. With the help of the equations (2.6) and (2.7), we get the identity.

Hence we have the Theorem.

3. Identities satisfied by Berwald's conformally changed curvature tensor $\bar{H}^{i}_{hjk}\,(x, \dot{x})$.

The conformally changed BERWALD curvature tensor is given by [4]

$$\left\{\begin{aligned}
\bar{H}^{i}_{hjk} &= H^{i}_{hjk} - 2\,\sigma_{m}\,\{ \dot{\partial}_{h}\, \partial_{[k}\, \dot{\partial}_{j]}\, B^{im} - \dot{\partial}_{h}\, \dot{\partial}_{\gamma}\, \dot{\partial}_{[j}\, B^{im} \\
&+ \dot{\partial}_{k]}\, G^{\gamma} - \dot{\partial}_{\gamma}\, \dot{\partial}_{[j}\, B^{im}\, G^{\gamma}_{k]h} + \dot{\partial}_{h}\, \dot{\partial}_{[j}\, B^{\gamma m}\, G^{i}_{k]\gamma} \\
&+ \dot{\partial}_{[j}\, B^{\gamma m}\, G^{i}_{k]\gamma h} \} + 2\, \dot{\partial}_{h}\, \dot{\partial}_{[k}\, B^{im}\, \partial_{j]}\, \sigma_{m} \\
&+ 2\, \sigma_{m}\, \sigma_{n}\, \{ \dot{\partial}_{h}\, \dot{\partial}_{[j}\, B^{\gamma n}\, \dot{\partial}_{k]}\, \dot{\partial}_{\gamma}\, B^{im} \\
&+ \dot{\partial}_{[j}\, B^{\gamma n}\, \dot{\partial}_{k]}\, \dot{\partial}_{h}\, \dot{\partial}_{\gamma}\, B^{im} \}.
\end{aligned}\right. \tag{3.1}$$

satisfies the following identities :

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(3.2)
$$\left\{\begin{aligned}
&\bar{H}^{i}_{hjk} - \bar{H}^{i}_{kjh} = H^{i}_{hjk} - H^{i}_{kjh} - 2\,\sigma_m\,\{\dot{\partial}_j\,\dot{\partial}_{[h}\,\delta_{k]}\,B^{im}\\
&+ \dot{\partial}_\gamma\,\dot{\partial}_j\,\dot{\delta}_{[k}\,B^{im}\,\dot{\partial}_{h]}\,G^\gamma + \dot{\partial}_\gamma\,\dot{\partial}_{[k}\,B^{im}\,G^\gamma_{h]j}\\
&+ \dot{\partial}_j\,\dot{\partial}_{[h}\,B^{\gamma m}\,G^i_{k]\gamma} + \dot{\partial}_{[h}\,B^{\gamma m}\,G^i_{k]\gamma j}\,\}\\
&+ 2\,\dot{\partial}_j\,\dot{\partial}_{[k}\,B^{im}\,\partial_{h]}\,\sigma_m + 2\,\sigma_m\,\sigma_n\,\{\dot{\partial}_j\,\dot{\partial}_{[h}\,B^{\gamma n}\,\dot{\partial}_{k]}\,\dot{\partial}_\gamma\,B^{im}\\
&+ \dot{\partial}_{[h}\,B^{\gamma n}\,\dot{\partial}_{k]}\,\dot{\partial}_j\,\dot{\partial}_\gamma\,B^{im}\},
\end{aligned}\right.$$

(3.3)
$$\left\{\begin{aligned}
&\bar{H}_{hljk} + \bar{H}_{lhkj} = e^{2\sigma}\,[\,H_{hljk} + H_{lhkj}\\
&- 4\,\sigma_m\,\{gi_{[l}\,\dot{\partial}_{h]}\,\partial_{[k}\,\dot{\partial}_{i]}\,B^{im} - gi_{\,[l}\,\dot{\partial}_{h]}\,\dot{\partial}_\gamma\,\dot{\partial}_{[j}\,B^{im}\,\dot{\partial}_{k]}\,{}^{\gamma}_{G}\\
&- \dot{\partial}_\gamma\,\dot{\partial}_{[j}\,B^{im}\,G^\gamma_{k]\,[h}\,g_{l]\,i}\,\} - 4\,\sigma m\,\{\,G_{[k\,[l\,\overline{|\gamma|}}\,\dot{\partial}_{j]}\,\dot{\partial}_{h]}\\
&- B^{\gamma m} + \dot{\partial}_{[j}\,B^{\gamma m}\,G^i_{k]\,\gamma\,[h}\,g_{l]\,i}\,\}\\
&+ 4\,gi_{[l}\,\dot{\partial}_{h]}\,\dot{\partial}_{[k}\,B^{im}\,\partial_{j]}\,\sigma m + 4\,\sigma m\,\sigma n\,\{gi_{\,[l}\,\dot{\partial}_{h]}\\
&+ \dot{\partial}_{[j}\,B^{\gamma n}\,\dot{\partial}_{k]}\,\dot{\partial}_\gamma\,B^{im} + \dot{\partial}_{[j}\,B^{\gamma n}\,\dot{\partial}_{k]}\,\dot{\partial}_\gamma\,\dot{\partial}_{[h}\,B^{im}\,g_{l]\,i}\,\}\,].
\end{aligned}\right.$$

(3.4)
$$\left\{\begin{aligned}
&\bar{H}_{hljk} - \bar{H}_{hkjl} + \bar{H}_{jkhl} - \bar{H}_{jlhk}\\
&= e^{2\sigma}\,[\,H_{hljk} - H_{hkjl} + H_{jkhl} - H_{jlhk}\\
&- 2\,\sigma m\,gil\,(\,\dot{\partial}_{[j}\,\partial_{h]}\,\dot{\partial}_k\,B^{im} + \dot{\partial}_\gamma\,\dot{\partial}_k\,\dot{\partial}_{[h}\,B^{im}\,\dot{\delta}_{j]}\,G^\gamma\\
&+ \dot{\partial}_\gamma\,\dot{\partial}_{[h}\,B^{im}\,G^\gamma_{j]k} + \dot{\partial}_{[j}\,B^{\gamma m}\,G^i_{h]\gamma k}\\
&- \dot{\partial}_k\,\dot{\partial}_{[j}\,B^{\gamma n}\,\dot{\partial}_{h]}\,\dot{\partial}_\gamma\,B^{im}\,\sigma n - \dot{\partial}_{[j}\,B^{\gamma n}\,\dot{\partial}_{h]}\,\dot{\partial}_k\,\dot{\partial}_\gamma\,B^{im}\,\sigma n)\\
&- 2\,\sigma m\,gik\,(\,\dot{\partial}_{[h}\,\partial_{j]}\,\dot{\partial}_l\,B^{im} + \dot{\partial}_\gamma\,\dot{\partial}_l\,\dot{\partial}_{[j}\,B^{im}\,\dot{\partial}_{h]}\,G^\gamma\\
&+ \dot{\partial}_\gamma\,\dot{\partial}_{[j}\,B^{im}\,G^\gamma_{h]l} + \dot{\partial}_{[h}\,B^{\gamma m}\,G^i_{j]\gamma l}\\
&- \dot{\partial}_l\,\dot{\partial}_{[h}\,B^{\gamma n}\,\dot{\partial}_{j]}\,\dot{\partial}_\gamma\,B^{im}\,\sigma n - \dot{\partial}_{[h}\,B^{\gamma n}\,\dot{\partial}_{j]}\,\dot{\partial}_l\,\dot{\partial}_\gamma\,B^{im}\,\sigma n)\\
&- 2\,\sigma m\,(\,\dot{\partial}_k\,\dot{\partial}_{[j}\,B^{\gamma m}\,G_{h]l\gamma} + \dot{\partial}_l\,\dot{\partial}_{[h}\,B^{\gamma m}\,G_{j]k\gamma}\,)\\
&+ 2\,(\,gil\,\dot{\partial}_k\,\dot{\partial}_{(h}\,B^{im}\,\partial_{j)}\,\sigma m + gik\,\dot{\partial}_l\,\dot{\partial}_{(i}\,B^{im}\,\partial_{h)}\,\sigma m\,)\,],
\end{aligned}\right.$$

$$
(3.5) \quad \left\{ \begin{aligned}
\bar{H}_{hljk} - \bar{H}_{lhkj} = e^{2\sigma} \, [\, & H_{hljk} - H_{lhkj} - 4\, \sigma m \\
& - \{ gi\,(l\, \dot{\partial}\, h)\ \partial_{(k}\, \dot{\partial}_{j)}\, B^{im} - gi\,(l\, \dot{\partial}\, h)\ \dot{\partial}_{\gamma}\, \dot{\partial}_{(j}\, B^{im}\, \dot{\partial}_{k)}\ G^{\gamma} \\
& - \dot{\partial}_{\gamma}\, \dot{\partial}_{(j}\, B^{im}\ G^{\gamma}_{k)}\ (h\, gl)\, i\,] - 4\, \sigma m\, \{ \dot{\partial}\, (h\ \dot{\partial}_{(j}\, B^{\gamma m}\ G_{k)}\ l)\, \gamma \\
& + \dot{\partial}_{(j} B^{\gamma m}\ G^{i}_{k)\,\gamma}\ (h\, gl)\, i \} + 4\, gi\,(l\, \dot{\partial} h)\ \dot{\partial}_{(k}\ b^{im}\ \partial_{j)}\ \sigma m \\
& + 4\, \sigma m\ \sigma n\ \{ gi\,(l\, \dot{\partial}\, h)\ \dot{\partial}_{(j}\, B^{\gamma n}\ \dot{\partial}_{k)}\ \dot{\partial}_{\gamma}\ B^{im} \\
& + \dot{\partial}_{(j}\, B^{\gamma n}\, \dot{\partial}_{k)}\ \dot{\partial}_{\gamma}\ \dot{\partial}\ (h\ B^{im}\ gl)\, i \}\,],
\end{aligned} \right.
$$

where

$$
(3.6) \qquad \bar{H}_{hljk} \overset{\text{def}}{=} \bar{g}_{il}\ \bar{H}^{i}_{hjk}\,.
$$

Proof. The proofs of these above identities follows the pattern of the proofs of theorems (2.1) and (2.2).

4. Conformal Invariants

Theorem 4.1 - *If* $X^i\,(x)$ *and* $X_i\,(x)$ *are the contravariant and covariant components of a vector* $X\,(x)$, *independent of the directional arguments, then the following quantities are conformal invariants* :

$$
N^{i}_{k}\ (x, \dot{x}) \overset{\text{def}}{=} X^{i}_{(k)} + \frac{1}{n}\ Y^{\gamma}_{\gamma h}\ (x, \dot{x})\ \dot{\partial}_{j}\ \dot{\partial}_{k}\ B^{ih}\ X^{j}
$$

and

$$
N^{*}_{ik}\ (x, \dot{x}) \overset{\text{def}}{=} X_{i\,(k)} - \frac{1}{n}\ Y^{\gamma}_{\gamma h}\ (x, \dot{x})\ \dot{\partial}_{i}\ \dot{\partial}_{k}\ B^{jh}\ X_{j}
$$

where (1) *denotes the covariant derivative in the sense of* BERWALD *and* $Y^{i}_{jk}\ (x, \dot{x})$ *are the* CHRISTOFFEL *symbols of the second kind* [3].

Proof. The CHRISTOFFEL symbols Y^{h}_{ij} under the conformal change are given [3] by:

$$
(4.1) \qquad \bar{Y}^{h}_{ij} = Y^{h}_{ij} + (\,\sigma_i\ \partial^{h}_{j} + \sigma_j\ \partial^{h}_{i} - g^{hk}\ g_{ij}\ \sigma_k\,)\,.
$$

Contracting (4.1) with respect to the indices h and i, we get

$$
(4.2) \qquad \bar{Y}^{\gamma}_{\gamma j} = Y^{\gamma}_{\gamma j} + n\ \sigma_j\ .
$$

The covariant derivative of $X^i\,(x)$ in the sense of BERWALD under the conformal change is

$$
(4.3) \qquad X^{i}_{(\bar{k})} = \frac{\delta X^i}{\delta x^k} + \bar{G}^{i}_{jk}\ (x, \dot{x})\ X^{j}\,.
$$

Hence we get

$$X^i_{(\overline{k})} - X^i_{(k)} = - X^j \, (G^i_{jk} - \bar{G}^i_{jk}) \, . \tag{4.4}$$

With the help of the equations (1.6) and (4.2) the equation (4.4) reduces to the form

$$X^i_{(\overline{k})} + \frac{1}{n} X^j \, \dot{\partial}_j \, \dot{\partial}_k \, B^{ih} \, \bar{Y}^\gamma_{\gamma h} = X^i_{(k)} + \frac{1}{n} X^j \, \dot{\partial}_j \, \dot{\partial}_k \, B^{ih} \, Y^\gamma_{\gamma h} \, . \tag{4.5}$$

Again, for the covariant component $X_i(x)$, we have

$$X_{i(\overline{k})} = \frac{\partial X_i}{\partial x^k} - X_j \, \bar{G}^j_{ik} \, (x, \dot{x}) \, . \tag{4.6}$$

Consequently, we get

$$X_{i(\overline{k})} - X_{i\,(k)} = X_j \, (G^j_{ik} - \bar{G}^j_{ik}). \tag{4.7}$$

In view of the equations (1.6) and (4.2) the equation (4.7) yields

$$X_{i\,(\overline{k})} - \frac{1}{n} X_j \, \bar{Y}^\gamma_{\gamma h} \, \dot{\partial}_i \, \dot{\partial}_k \, B^{jh} = X_{i\,(k)} - \frac{1}{n} X_j \, Y^\gamma_{\gamma h} \, \dot{\partial}_i \, \dot{\partial}_k \, B^{jh} \, . \tag{4.8}$$

Hence we have the Theorem.

Theorem 4.2— If $X^i(x, \dot{x})$ and $X_i(x, \dot{x})$ are the contravariant and covariant components of a vector $X(x, \dot{x})$ then the rollowing quantities are conformal invariants

$$M^i_k \, (x, \dot{x}) \stackrel{\text{def}}{=} X^i_{(k)} - \frac{1}{n} Y^\gamma_{\gamma l} \, (x, \dot{x}) \, [\, \dot{\partial}_h \, X^i \, \dot{\partial}_k \, B^{hl} - X^h \, \dot{\partial}_h \, \dot{\partial}_k \, B^{il} \,]$$

and

$$M^*_{ik} \, (x, \dot{x}) \stackrel{\text{def}}{=} X_{i\,(k)} - \frac{1}{n} Y^\gamma_{\gamma l} \, [\, \dot{\partial}_h \, X_i \, \dot{\partial}_k \, B^{hl} + X_h \, \dot{\partial}_i \, \dot{\partial}_k \, B^{hl} \,].$$

Proof. The covariant derivative of $X^i(x, x)$ in the sense of BERWALD under the conformal change is given by

$$X^i_{(\overline{k})} = \frac{\partial X^i}{\partial x^k} - \frac{\partial X^i}{\partial \dot{x}^h} \, \frac{\partial \bar{G}^h}{\partial \dot{x}^k} + \bar{G}^i_{kh} \, (x, \dot{x}) \, X^h \, . \tag{4.9}$$

Therefore we get

$$X^i_{(\overline{k})} - X^i_{(k)} = \frac{\partial X^i}{\partial \dot{x}^h} \left(\frac{\partial \, G^h}{\partial \dot{x}^k} - \frac{\partial \bar{G}^h}{\partial \dot{x}^k} \right) + X^h \, (\bar{G}^i_{hk} - G^i_{hk}). \tag{4.10}$$

With the help of the equations (1.6) and (4.2) the equation (4.10) reduces to the form

$$X^i_{(\overline{k})} - X^i_{(k)} = \frac{1}{n} \, \frac{\partial \, X^i}{\partial \dot{x}^h} \, \dot{\partial}_k \, B^{hl} \, (\bar{Y}^\gamma_{\gamma l} - Y^\gamma_{\gamma l}) - \frac{1}{n} \, X^h \, \dot{\partial}_h \, \dot{\partial}_k \, B^{il} \, (\bar{Y}^\gamma_{\gamma l} - Y^\gamma_{\gamma l}), \tag{4.11}$$

which yields the form

$$(4.12)\qquad \begin{aligned} & X^i_{(\bar{k})} - \frac{1}{n}\,\bar{Y}^{\gamma}_{\gamma l}\left[\frac{\partial X^i}{\partial \dot{x}^h}\,\dot{\partial}_k\,B^{hl} - X^h\,\dot{\partial}_h\,\dot{\partial}_k\,B^{il}\right] \\ & = X^i_{(k)} - \frac{1}{n}\,Y^{\gamma}_{\gamma l}\left[\frac{\partial X^i}{\partial \dot{x}^h}\,\dot{\partial}_k\,B^{hl} - X^h\,\dot{\partial}_h\,\dot{\partial}_k\,B^{il}\right]. \end{aligned}$$

Again, we have

$$(4.13)\qquad X_{i(\bar{k})} = \frac{\partial X_i}{\partial x^k} - \frac{\partial X_i}{\partial \dot{x}^h}\,\frac{\partial \bar{G}^h}{\partial \dot{x}^k} - X_h\,\bar{G}^h_{ik}\,.$$

Hence we get

$$(4.14)\qquad X_{i(\bar{k})} - X_{i(k)} = \frac{\partial X_i}{\partial \dot{x}^h}\,(\dot{\partial}_k\,G^h - \dot{\partial}_k\,\bar{G}^h) + X_h\,(G^h_{ik} - \bar{G}^h_{ik})\,.$$

With the help of the equations (1.6) and (4.2) it becomes

$$(4.15)\qquad \begin{aligned} X_{i(\bar{k})} - X_{i(k)} &= \frac{1}{n}\,\dot{\partial}_h\,X_i\,\dot{\partial}_k\,B^{hl}\,(\bar{Y}^{\gamma}_{\gamma l} - Y^{\gamma}_{\gamma l}) \\ &+ \frac{1}{n}\,X_h\,\dot{\partial}_i\,\dot{\partial}_k\,B^{hl}\,(\bar{Y}^{\gamma}_{\gamma l} - Y^{\gamma}_{\gamma l})\,. \end{aligned}$$

Hence we have the Theorem.

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ÖZET

FINSLER uzayında konform tasvirler M. S. KNEBELMAN [1], H. HOMBU [2], H. RUND [3] ve R. S. SINHA [4] tarafından incelenmiştir. Bu yazıda ayni konu ele alınarak eğrilik tensörlerinin konform dönüştürülmüşleri için ifadeler verilmiş ve bunların sağladıkları bazı özdeşlikler bulunmuştur.