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Existence of Solution to Fractional Order Impulsive Partial Hyperbolic Differential Equations with Infinite Delay

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Abstract

In this article, we investigate the existence of solutions to a class of initial value problem (for short IVP) for fractional order impulsive partial hyperbolic differential equations (for short FOIPHDEs) with infinite delay. Here we use Mixed Riemann-Liouville fractional derivative to construct the considered FOIPHDEs. The analysis of this article is based on Burton-Kirk fixed point theorem. A new existence result for FOIPHDEs with infinite delay has been obtained. To support the analytic proof, we give an illustrative example.

Keywords: FOIPHDEs, Burton-Kirk fixed point theorem.
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1. Introduction

The purpose of this study is to establish an existence criterion of solutions to the following class of IVP for FOIPHDEs with infinite delay:

$$\left(D_{z_k}^r u\right)(x,\,y) = f\left(x, y, u_{(x,y)}\right); if\,(x,y) \in J_k,\ k = 0,1,2,\cdots,m, \tag{1}$$

$$u\left(x_k^+,\,y\right) = u\left(x_k^-,\,y\right) + I_k\left(u\left(x_k^-,\,y\right)\right); if\, y \in [0,\,b]\,,\ k = 1,2,\cdots,m, \tag{2}$$

$$u\left(x,\,y\right) = \phi\left(x,\,y\right); if\,(x,\,y) \in \tilde{J}, \tag{3}$$

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$$u(x, 0) = \varphi(x), \ x \in [0, a] \quad u(0, y) = \psi(y), \ y \in [0, b], \tag{4}$$

where, $D_{z_k}^r$ is the mixed Riemann-Liouville fractional differential operator of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $J_0 = [0, x_1] \times [0, b]$, $J_k = (x_k, x_{k+1}) \times [0, b]$; $k = 1, 2, \cdots, m$, $z_k = (x_k, 0)$; $k = 0, 1, 2, \cdots, m$, $a, b > 0$, $J = [0, a] \times [0, b]$, $\tilde{J} = (-\infty, a] \times (-\infty, b] \setminus (0, a] \times (0, b]$, $\varphi : [0, a] \to \mathbf{R}^n$, $\psi : [0, b] \to \mathbf{R}^n$ are given continuous functions with $\varphi(x) = \phi(x, 0)$, $\psi(y) = \phi(0, y)$ for each $(x, y) \in J$, $0 = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = a$, $f : J \times B \to \mathbf{R}^n$, $I_k : \mathbf{R}^n \to \mathbf{R}^n$, $k = 1, 2, \cdots, m$, $\phi : \tilde{J} \to \mathbf{R}^n$ are given functions and B is a phase space which will be specified in the next section. If $u : (-\infty, a] \times (-\infty, b] \to \mathbf{R}^n$ then for any $(x, y) \in J$, $u_{(x,y)}$ is defined by

$$u_{(x, y)}(s, t) = u(x + s, y + t) \ for \ (s, t) \in [-\alpha, 0] \times [-\beta, 0].$$

The necessity of fractional order differential equations (for short FDEs) lies in the fact that fractional order model is more accurate than integer order models, that is, there is more degree of freedom in the fractional order models. Furthermore, fractional order derivatives provide an excellent mechanism for the description of memory and hereditary properties of various materials and processes. In applied sense, FDEs arise in various engineering and scientific disciplines for mathematical modeling in the fields of physics, chemistry, biology, fluid flow, electromagnetic theory, polymer rheology, electrical network, statistics, economics, signal and image processing, viscoelasticity, aerodynamics and porous media, etc., see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and their cited references. Some recent development of ordinary and partial fractional differential equations can be found in the monographs of Abbas *et al.* [11], Kilbas *et al.* [8], Podlubny [10], the papers of Agarwal *et al.* [12], Asaduzzaman and Zulfikar Ali [13], Zhu *et al.* [14], Zhang and Fu [15], Agarwal *et al.* [16], Hemeda [17], Abbas *et al.* [18, 19], Abbas and Benchohra [20, 21, 22], Agarwal *et al.* [23], Benchohra *et al.* [24], Benchohra and Slimani [25], Vityuk and Golushkov [26] and the references therein.

Initial value problems for FOIPHDEs have been addressed by several researchers during last few decades. In current literature, some researchers have been studied the existence of solutions of initial value problems for FOIPHDEs, see for instance [27, 28, 29, 30, 31] and their cited references. Theory of functional differential equations is a significant branch of nonlinear analysis. Functional differential equations or differential delay equations have been used in modeling of different scientific phenomena for long time. Frequently, it has been supposed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay, see for instance [32, 33, 34, 35, 36, 37, 38, 39].

On the other hand, theory of impulsive differential equations has become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see for instance [24, 40] and their cited references.

To the best of our knowledge, there is no any work considering the existence of solutions to the initial value problem given by (1) to (4), using Burton-Kirk fixed point theorem [41, 42]. Therefore, our main object is to establish the existence criteria of solutions to the initial value problem for FOIPHDEs given by (1) to (4), using Burton-Kirk fixed point theorem. The rest of this work is furnished as follows: In Section 2, we provide some basic definitions, lemmas and state Burton-Kirk fixed point theorem. Section 3 is used to state and prove our main results, which provide us a technique to check the existence of at least one solutions of initial value problem for FOIPHDEs given by (1) to (4). Finally, in Section 4, we give an example to verify our main result.

2. Preliminaries Notes

In this section, we introduce some necessary definitions and preliminary facts which will be used throughout this paper. First, we introduce phase space. The phase space plays an important role in the study of both qualitative and quantitative theory of functional differential equations. To define this space, we usually choose a semi-normed space satisfying some suitable axioms, which was presented by Hale and Kato [38] (see [32, 33, 35]) for functional ordinary differential equations.

Definition 2.1. *(see [32, 33, 35, 38]). For any $(x, y) \in J$, we denote*

$$E_{(x, y)} = [0, x] \times \{0\} \bigcup \{0\} \times [0, y], \tag{5}$$

furthermore for $x = a$, $y = b$ we write simply E. The phase space $(B, \|., .\|)$ is a semi-normed linear space of functionals mapping from $(-\infty, 0] \times (-\infty, 0]$ into $\boldsymbol{R}^n$ satisfying the following fundamental axioms:
(I) If $z : (-\infty, a] \times (-\infty, b] \to \boldsymbol{R}^n$ and $z_{(x, y)} \in B$; for all $(x, y) \in E$, then there are constants $H, K, M > 0$ such that for any $(x, y) \in J$, the following conditions hold:
(i) $z_{(x, y)} \in B$,
(ii) $\|z(x, y)\| \leq H\|z_{(x, y)}\|_B$,
(iii) $\|z_{(x, y)}\|_B \leq K \sup_{(s, t) \in [0, x] \times [0, y]} \|z(s, t)\| + M \sup_{(s, t) \in E_{(x, y)}} \|z_{(s, t)}\|_B$,
(II) The space B is complete.

Example 2.2. *If the phase space B is a set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \boldsymbol{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$, $\alpha, \beta \geq 0$, with the semi-norm*

$$\|\phi\|_B = \sup_{(s, t) \in [-\alpha, 0] \times [-\beta, 0]} \|\phi(s, t)\|, \tag{6}$$

then we have $H = K = M = 1$.

Throughout this paper, $L^1(J, \mathbf{R}^n)$ denote the space of Lebesgue-integrable functions $u : J \to \mathbf{R}^n$ with the norm $\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y,)\| dxdy$ and $C(J, \mathbf{R}^n)$ denote the space of continuous functions $u : J \to \mathbf{R}^n$ with the norm $\|u\|_\infty = \sup_{(x, y) \in J} \|u(x, y)\|$.
Among the different definitions of partial fractional derivative and partial fractional integral, the most frequent used definitions are Riemann-Liouville partial fractional integral, Riemann-Liouville partial fractional derivative and Caputo partial fractional derivative, see for instance [2, 8, 10, 11].

Definition 2.3. *(see [4, 8, 9]). Let $\alpha \in (0, \infty)$ and $u \in L^1(J, \boldsymbol{R}^n)$. The Riemann-Liouville partial fractional integral of order α of $u(x, y)$ with respect to x is defined by*

$$I_{0, x}^\alpha u(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha - 1} u(s, y) ds,$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$, where $\Gamma(\alpha)$ is the Euler Gamma function of α and provided that the integral exists.
Similarly, the Riemann-Liouville partial fractional integral of order α of $u(x, y)$ with respect to y is defined by

$$I_{0, y}^\alpha u(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y - s)^{\alpha - 1} u(x, s) ds,$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$.

Definition 2.4. *(see [4, 8, 9]). Let $\alpha \in (0, \infty)$ and $u \in L^1(J, \boldsymbol{R}^n)$. The Riemann-Liouville partial fractional derivative of order α of $u(x, y)$ with respect to x is defined by*

$$\left(D_{0, x}^\alpha u\right)(x, y) = \frac{\partial}{\partial x} \left(I_{0, x}^{1-\alpha} u(x, y)\right),$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$.
Similarly, the Riemann-Liouville partial fractional derivative of order α of $u(x, y)$ with respect to y is defined by

$$\left(D_{0, y}^\alpha u\right)(x, y) = \frac{\partial}{\partial x} \left(I_{0, y}^{1-\alpha} u(x, y)\right),$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$.

Definition 2.5. *(see [4, 8, 9]). Let $\alpha \in (0, \infty)$ and $u \in L^1(J, \mathbf{R}^n)$. The Caputo partial fractional derivative of order α of $u(x, y)$ with respect to x is defined by*

$$\left({}^C D_{0,\,x}^\alpha u\right)(x, y) = I_{0,\,x}^{1-\alpha}\left(\frac{\partial}{\partial x} u(x, y)\right),$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$.
Similarly, the partial Caputo fractional derivative of order α of $u(x, y)$ with respect to y is defined by

$$\left({}^C D_{0,\,y}^\alpha u\right)(x, y) = I_{0,\,y}^{1-\alpha}\left(\frac{\partial}{\partial x} u(x, y)\right),$$

for almost all $x \in [0, a]$ and for almost all $y \in [0, b]$.

In general, partial Caputo fractional derivative and partial Riemann -Liouville fractional derivative of a function are not same. In particular, the solution space of ${}^C D_{0,\,y}^\alpha u(t) = 0$ is spanned by $\left\{1, t, t^2, \cdots, t^{n-1}\right\}$, while the solution space of $D_{0,\,y}^\alpha u(t) = 0$ is spanned by $\left\{t^{\alpha-1}, t^{\alpha-2}, \cdots, t^{\alpha-n}\right\}$.

Definition 2.6. *(see [8, 26]). Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $z_k = (x_k, 0)$ and $u \in L^1(J, \mathbf{R}^n)$. The left-sided mixed Riemann-Liouville fractional integral of order r of $u(x, y)$ is defined by*

$$\left(I_{z_k+}^r u\right)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} u(s, t)\, dt\, ds, \tag{7}$$

and the right-sided mixed Riemann-Liouville fractional integral of order r of $u(x, y)$ is defined by

$$\left(I_{z_k-}^r u\right)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_x^{x_k} \int_y^0 (s - x)^{r_1-1}(t - y)^{r_2-1} u(s, t)\, dt\, ds, \tag{8}$$

where $\Gamma(r_1)\Gamma(r_2)$ are Euler Gamma function of r_1, r_2 respectively and provided that the integral exists.

Definition 2.7. *(see [8, 26]). Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $z_k = (x_k, 0)$ and $u \in L^1(J, \mathbf{R}^n)$. Then the left-sided mixed Riemann-Liouville fractional derivative of order r of u is defined by*

$$\left(D_{z_k+}^r u\right)(x, y) = \frac{\partial^2}{\partial x \partial y}\left(I_{z_k+}^{1-r} u\right)(x, y), \tag{9}$$

and right-sided mixed Riemann-Liouville fractional derivative of order r of u is defined by

$$\left(D_{z_k-}^r u\right)(x, y) = \frac{\partial^2}{\partial x \partial y}\left(I_{z_k-}^{1-r} u\right)(x, y), \tag{10}$$

and left-sided mixed Caputo fractional derivative of order r of u is defined by

$$\left({}^C D_{z_k+}^r u\right)(x, y) = \left(I_{z_k+}^{1-r} \frac{\partial^2}{\partial x \partial y} u\right)(x, y), \tag{11}$$

and right-sided mixed Caputo fractional derivative of order r of u is defined by

$$\left({}^C D_{z_k-}^r u\right)(x, y) = \left(I_{z_k-}^{1-r} \frac{\partial^2}{\partial x \partial y} u\right)(x, y), \tag{12}$$

where $1 - r$ mean that $(1 - r_1, 1 - r_2) \in (0, \infty) \times (0, \infty)$ and $D_{xy}^2 = \dfrac{\partial^2}{\partial x \partial y}$ denote the mixed second order partial derivative.

To establish the main result, we need the following generalization of GronwallâĂŹs lemma for two independent variables and singular kernel.

Lemma 2.8. *(see [43]). Let $v : J \to [0, \infty)$ be a real function and $w(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$, such that*

$$v(x, y) \leqslant w(x, y) + c \int_0^x \int_0^y \frac{v(s, t)}{(x - s)^{r_1} (y - t)^{r_2}} dt ds, \tag{13}$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$v(x, y) \leqslant w(x, y) + \delta c \int_0^x \int_0^y \frac{w(s, t)}{(x - s)^{r_1} (y - t)^{r_2}} dt ds, \tag{14}$$

for every $(x, y) \in J$.

Theorem 2.9. *(Burton-Kirk fixed point theorem) (see [41, 42]). Let $(X, \|.\|)$ be a Banach space, and $S, T : X \to X$ two operators satisfying:*
(i) S is completely continuous, and
(ii) T is contraction with contraction constant $\alpha < 1$.
Then either
(a) the operator equation $u = \lambda T(u/\lambda) + \lambda S(u)$ has a solution in X for $\lambda = 1$, or
(b) the set $P = \{u \in X : u = \lambda T(u/\lambda) + \lambda S(u)\}$ of all such solutions is unbounded for $0 < \lambda < 1$.

Now, we define the solutions of our problem given by (1) to (4) and for these solutions we shall consider the space

$$\Omega = \{u : (-\infty, a] \times (-\infty, b] \to \mathbf{R}^n : u_{(x,y)} \in B \, for \, (x, y) \in E \, and$$
$$\exists \, u(x_k^-, .), u(x_k^+, .) \; exist \, with \; u(x_k^-, .) = (x_k, .);$$
$$k = 1, 2, \cdots, m \; and \; u \in C(J_k, \mathbf{R}^n); k = 0, 1, \cdots, m, \}$$

where $J_k = (x_k, x_{k+1}] \times (0, b]$. Let

$$J' = J \setminus \{(x_1, y), (x_2, y), \cdots, (x_m, y), y \in [0, b]\}.$$

For $u \in \Omega$, we define the function $\tilde{u}_k \in C([x_k, x_{k+1}] \times [0, b], \mathbf{R}^n)$ by

$$\tilde{u}_k(x, y) = u(x, y) \; for \; (x, y) \in [x_k, x_{k+1}] \times [0, b]$$

and

$$\tilde{u}_k(x_k, y) = \lim_{k \to x_k^+} u(x, y), \; for \, k = 0, 1, \cdots, m.$$

Furthermore, for a set $D \subset \Omega$, we represent the set $\tilde{D}_k$ by $\tilde{D}_k = \{\tilde{u}_k : u \in D\}$, for $k = 0, 1, \cdots, m$.

Lemma 2.10. *(see [44]). A set $D \subset \Omega$ is relatively compact if and only if, each set $\tilde{D}_k$, for $k = 0, 1, \cdots, m$, is relatively compact in $C([x_k, x_{k+1}] \times [0, b], \mathbf{R}^n)$.*

Definition 2.11. *A function $u \in \Omega$ is said to a solution of the IVP given by (1) to (4) if u satisfies $(D_0^r(x, y)) = f(x, y, u(x,))$ on J'.*

Let us consider $h \in C([x_k, x_{k+1}] \times [0, b], \mathbf{R}^n)$, $z_k = (x_k, 0)$ and

$$\mu_k(x, y) = u(x, 0) + u(x_k^+, y) - u(x_k^+, 0), \; k = 0, 1, \cdots, m. \tag{15}$$

The following lemma will be needed to establish the existence of solutions of our problem given by (1) to (4):

Lemma 2.12. *A function* $u \in C\left([x_k, x_{k+1}] \times [0, b], \mathbf{R}^n\right)$, $k = 0, 1, \cdots, m$ *is a solution of following partial fractional differential equation*

$$\left(D^r_{z_k} u\right)(x, y) = h(x, y); \ (x, y) \in [x_k, x_{k+1}] \times [0, b], \tag{16}$$

if and only if $u(x, y)$ *satisfies*

$$u(x, y) = \mu_k(x, y) + \left(I^r_{z_k} h\right)(x, y); \ (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{17}$$

Proof. Let $u(x, y)$ be a solution of $\left(D^r_{z_k} u\right)(x, y) = h(x, y); \ (x, y) \in [x_k, x_{k+1}] \times [0, b]$. Then from the definition of the derivative $(D^r_{z_k} u)(x, y)$, we have

$$\frac{\partial^2}{\partial x \partial y}\left(I^{1-r}_{z_k} u\right)(x, y) = h(x, y). \tag{18}$$

Hence, we obtain that

$$I^r_{z_k}\left(\frac{\partial^2}{\partial x \partial y}\left(I^{1-r}_{z_k} u\right)\right)(x, y) = \left(I^r_{z_k} h\right)(x, y), \tag{19}$$

which gives us

$$I^1_{z_k}\left(\frac{\partial^2}{\partial x \partial y} u\right)(x, y) = \left(I^r_{z_k} h\right)(x, y). \tag{20}$$

But we know that

$$I^1_{z_k}\left(\frac{\partial^2}{\partial x \partial y} u\right)(x, y) = u(x, y) - u(x, 0) - u\left(x_k^+, y\right) + u\left(x_k^+, 0\right).$$

So from (20), we yield that

$$u(x, y) = \mu_k(x, y) + \left(I^r_{z_k} h\right)(x, y), \ \forall (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{21}$$

Conversely, suppose that $u(x, y)$ satisfy (17). Then, it is obvious that $u(x, y)$ must be a solution of $\left(D^r_{z_k} u\right)(x, y) = h(x, y)$, for all $(x, y) \in [x_k, x_{k+1}] \times [0, b]$. This completes the proof. $\qquad \square$

Lemma 2.13. *Let* $0 < r_1, r_2 \le 1$ *and let* $h : J \to \mathbf{R}^n$ *be a continuous function. Then the function* $u(x, y)$ *is a solution of the fractional integral equation*

$$u(x, y) = \begin{cases} \phi(x, y) & \text{if } (x, y) \in \tilde{J}, \\ \mu(x, y) + \sum\limits_{0 < x_k < x}\left(I_k\left(u\left(x_k^-, y\right)\right) - I_k\left(u\left(x_k^-, 0\right)\right)\right) & \text{if } (x, y) \in J, \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum\limits_{0 < x_k < x}\int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1 - 1}(y - t)^{r_2 - 1} h(s, t)\, dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} h(s, t)\, dtds, \ k = 1, 2, \cdots, m, \end{cases} \tag{22}$$

if and only if $u(x, y)$ *is a solution of the fractional IVP*

$$\left(D^r_{z_k} u\right)(x, y) = h(x, y); \ (x, y) \in J_k, k = 0, 1, \cdots, m, \tag{23}$$

$$u\left(x_k^+, y\right) = u\left(x_k^-, y\right) + I_k u\left(x_k^-, y\right), \ k = 1, 2, \cdots, m. \tag{24}$$

Proof. The proof of this lemma follows from the Lemma 2.12. $\qquad \square$

3. Main Results

This section is devoted to establish the existence criteria for solutions to a class of initial value problem for fractional order impulsive partial hyperbolic differential equations with infinite delay given by (1) to (4) applying a fixed point theorem (Theorem 2.9) due to Burton and Kirk [41, 42]. To establish the desired existence criteria, we need the following assumptions:

(A_1) The functions $I_k : \mathbf{R}^n \to \mathbf{R}^n$ and $f : J \times B \to \mathbf{R}^n$ are continuous.

(A_2) There exists $p, q \in C(J, \mathbf{R}_+)$ such that

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_B \; for \, (t,x) \in J \; and \, u \in B.$$

(A_3) There exists $l > 0$ such that

$$\|I_k(u) - I_k(v)\| \le l\|u - v\| \; for \, each \, u, v \in \mathbf{R}^n.$$

We are now in position to present and prove our main results.

Theorem 3.1. *Suppose that the assumptions (A_1), (A_2) and (A_3) are hold. If there exists $l > 0$ such that*

$$2ml < 1, \tag{25}$$

then the initial value problem given by (1) to (4) has least one solution on J .

Proof. We shall diminish the existence of solutions of IVP given by (1) to (4) to a fixed point problem. Consider an operator $A : \Omega \to \Omega$ which is defined as follows:

$$A(u)(x,y) = \begin{cases} \phi(x,y) & if \, (x,y) \in \tilde{J}, \\ \mu(x,y) + \sum\limits_{0<x_k<x} \left(I_k\left(u\left(x_k^-,y\right)\right) - I_k\left(u\left(x_k^-,0\right)\right)\right) & if \, (x,y) \in J, \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) \, dt ds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t) \, dt ds, \; k=1,2,\cdots,m, \end{cases}$$

Now, we set two operators $S, T : \Omega \to \Omega$, which are defined in the following way:

$$S(u)(x,y) = \begin{cases} \phi(x,y) & if \, (x,y) \in \tilde{J}, \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1} & if \, (x,y) \in J, \\ f(s,t,u(s,t)) dt ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ f(s,t,u(s,t)) dt ds, & k=1,2,\cdots,m, \end{cases}$$

and

$$T(u)(x,y) = \begin{cases} 0, & if \, (x,y) \in \tilde{J}, \\ \mu(x,y) & if \, (x,y) \in J, \\ + \sum\limits_{0<x_k<x} \left(I_k\left(u\left(x_k^-,y\right)\right) - I_k\left(u\left(x_k^-,0\right)\right)\right), & k=1,2,\cdots,m. \end{cases}$$

Let $v : (-\infty, a] \times (-\infty, b] \to \mathbf{R}^n$ be a function defined by

$$v(x,y) = \begin{cases} \phi(x,y), & if \, (x,y) \in \tilde{J}, \\ \mu(x,y), & if \, (x,y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in E$.

Now for each $w \in (J, \mathbf{R}^n)$ with $w(x,y) = 0$ for every $(x,y) \in E$, we define a function $\bar{w}$ by

$$\bar{w}(x,y) = \begin{cases} 0, & if \, (x,y) \in \tilde{J}, \\ w(x,y), & if \, (x,y) \in J. \end{cases}$$

If u satisfies the integral equation,

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)}) dt ds, \tag{26}$$

then we can decompose u as $u(x,y) = \bar{w}(x,y) + v(x,y)$; $(x,y) \in (x_k, x_{k+1}] \times [0,b]$, which implies that $u_{(x,y)} = \bar{w}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J \times [0,b]$, and the function w satisfies

$$\begin{aligned}
w(x,y) = &\sum_{0<x_k<x} \left(I_k\left(u\left(x_k^-, y\right)\right) - I_k\left(u\left(x_k^-, 0\right)\right)\right) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\bar{w}_{(s,t)} + v_{(s,t)}) dt ds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\bar{w}_{(s,t)} + v_{(s,t)}) dt ds.
\end{aligned} \tag{27}$$

Now, if we set

$$B_0 = \{w \in \Omega : w(x,y) = 0 \ for \ (x,y) \in E\},$$

and let $\|\cdot\|_{B_0}$ be the norm in B_0, which is defined by

$$\|w\|_{B_0} = \sup_{(x,y)\in E} \|w_{(x,y)}\|_B + \sup_{(x,y)\in J} \|w(x,y)\| = \sup_{(x,y)\in J} \|w(x,y)\|, \ w \in B_0,$$

then it is clear that B_0 is a Banach space with norm $\|\cdot\|_{B_0}$. If we consider two operators $S, T : B_0 \to B_0$, which are defined in the following way:

$$S(w)(x,y) = \begin{cases} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum\limits_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \\ f(s,t,\bar{w}_{(s,t)} + v_{(s,t)}) dt ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1} \\ (y-t)^{r_2-1} f(s,t,\bar{w}_{(s,t)} + v_{(s,t)}) dt ds, \\ \text{where } (x,y) \in J, \ k = 1,2,\cdots,m, \end{cases}$$

and

$$T(w)(x,y) = \mu(x,y) + \sum_{0<x_k<x} \left(I_k\left(u\left(x_k^-, y\right)\right) - I_k\left(u\left(x_k^-, 0\right)\right)\right),$$

$$\text{where } (x,y) \in J, \ k = 1,2,\cdots,m,$$

then the problem of finding solutions of the IVP given by (1) to (4) is diminished to finding solutions of the operator equation $S(w) + T(w) = w$. To prove this theorem, we shall prove that the operators S and T satisfy all the conditions of Theorem 2.9. The proof will be completed in the following steps.

Step-1: In this step we prove that the operator S is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ as $n \to \infty$ in B_0, then for each $(x,y) \in J$ we have

$$\begin{aligned}
&\|S(w_n)(x,y) - S(w)(x,y)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \\
&\|f(s,t,\bar{w}_{n(s,t)} + v_{n(s,t)}) - f(s,t,\bar{w}_{(s,t)} + v_{(s,t)})\| dt ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y \\
&(x-s)^{r_1-1}(y-t)^{r_2-1} |f(s,t,\bar{w}_{n(s,t)} + v_{n(s,t)}) - f(s,t,\bar{w}_{(s,t)} + v_{(s,t)})\| dt ds \\
&\leq \frac{\|f(.,.,\bar{w}_{n(.,.)} + v_{n(.,.)}) - f(.,.,\bar{w}_{(.,.)} + v_{(.,.)})\|}{\Gamma(r_1)\Gamma(r_2)} \left[\sum_{0<x_k<x} \left(\int_{x_{k-1}}^{x_k} \int_0^y\right.\right. \\
&\left.\left. (x_k - s)^{r_1-1}(y-t)^{r_2-1} dt ds\right) + \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds\right].
\end{aligned}$$

Since, f is continuous function, then we have

$$\|S(w_n) - S(w)\|_{B_0} \leq \frac{2a^{r_1}b^{r_2}\|f(.,.,\bar{w}_{n(.,.)}) - f(.,.,\bar{w}_{(.,.)})\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0, \, as \, n \to \infty. \tag{28}$$

Therefore, the operator S is continuous.

Step-2: In this step we prove that the operator S maps on bounded sets in B_0.

To complete this step, it is sufficient to prove that for any m^*, there exists a positive constant τ such that, $\|S(w)\|_{B_0} \leq \tau$ for every $w \in B_{m^*} = \{w \in B_0 : \|w\|_{B_0} \leq m^*\}$.

From assumption (A_2) we have for each $(x,y) \in (x_k, x_{k+1}] \times [0,b]$,

$$\|S(w)(x,y)\|$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})\|dtds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})\|dtds$$

$$\leq \frac{\|p\|_\infty + \|q\|_\infty m^*}{\Gamma(r_1)\Gamma(r_2)} \left[\sum_{k=1}^m \left(\int_{x_{k-1}}^{x_k} \int_0^y (x_k-s)^{r_1-1}(y-t)^{r_2-1}dtds\right)\right.$$

$$\left. + \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\right].$$

Hence

$$\|S(w)\|_B \leq \frac{2a^{r_1}b^{r_2}(\|p\|_\infty + \|q\|_\infty m^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} = \tau \, (say), \tag{29}$$

where

$$\|\bar{w}_{(s,t)} + v_{(s,t)}\|_B \leq \|\bar{w}_{(s,t)}\|_B + \|v_{(s,t)}\|_B \leq Km^* + K\|\phi(0,0)\| + M\|\phi\|_B = \xi \, (say).$$

Therefore,

$$\|S(w)\|_{B_0} \leq \tau.$$

Step-3: In this step we prove that the operator S maps from bounded sets into equicontinuous in B_0.

Let $(x_1,y_1),(x_2,y_2) \in (0,a] \times (0,b]$, $x_1 < x_2$, $y_1 < y_2$ and B_{m^*} be a bounded set as in the step-2. Now for any $w \in B_{m^*}$, we have

$$\|S(w)(x_2,y_2) - S(w)(x_1,y_1)\|$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k-s)^{r_1-1}\left[(y_2-t)^{r_2-1} - (y_1-t)^{r_2-1}\right]$$

$$f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})dtds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k-s)^{r_1-1}(y_2-t)^{r_2-1}$$

$$\|f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})\|dtds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} \left[(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\right.$$

$$\left. - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}\right] f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})dtds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})\|dtds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,\bar{w}_{(s,t)}+v_{(s,t)})\|dtds$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} \|f(s, t, \bar{w}_{(s,t)} + v_{(s,t)})\| dt ds$$

$$\leq \frac{\|p\|_\infty + \|q\|_\infty m^*}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \left[\sum_{k=1}^m \left(\int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1 - 1} \left[(y_2 - t)^{r_2 - 1} \right. \right. \right.$$

$$\left. - (y_1 - t)^{r_2 - 1} \right] dt ds \Big) + \sum_{k=1}^m \left(\int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} dt ds \right)$$

$$+ \int_0^{x_1} \int_0^{y_1} \left[(x_2 - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} - (x_1 - s)^{r_1 - 1} (y_1 - t)^{r_2 - 1} \right] dt ds$$

$$+ \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} dt ds$$

$$+ \int_0^{x_1} \int_{y_1}^{y_2} (x_2 - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} dt ds$$

$$+ \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} dt ds \Bigg]$$

$$\leq \frac{\|p\|_\infty + \|q\|_\infty m^*}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \left[\sum_{k=1}^m \left(\int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1 - 1} \left[(y_2 - t)^{r_2 - 1} \right. \right. \right.$$

$$\left. - (y_1 - t)^{r_2 - 1} \right] dt ds \Big) + \sum_{k=1}^m \left(\int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1 - 1} (y_2 - t)^{r_2 - 1} dt ds \right)$$

$$+ (2y_2^{r_2} (x_2 - x_1)^{r_1} + 2x_2^{r_1} (y_2 - y_1)^{r_2} + x_1^{r_1} y_1^{r_2} - x_2^{r_1} y_2^{r_2} - 2(x_2 - x_1)^{r_1} (y_2 - y_1)^{r_2}) \Big]$$

$$\to 0 \text{ as } x_1 \to x_2 \text{ and } y_1 \to y_2.$$

Hence, the operator S maps from bounded sets into equicontinuous in B_0.

Combining the consequences of step-1 to step-3 and applying the Arzela-Ascoli theorem [45], we can accomplish that the operator $S : B_0 \to B_0$ is completely continuous.

Step-4: In this step we prove that the operator T is contraction. Let $w, w^* \in B_0$, then for each $(x, y) \in J$ we have

$$\|T(w)(x, y) - T(w^*)(x, y)\|$$

$$\leq \sum_{k=1}^m \left(\|I_k(w(x_k^-, y)) - I_k(w^*(x_k^-, y))\| + \|I_k(w(x_k^-, 0)) - I_k(w^*(x_k^-, 0))\| \right)$$

$$\leq \sum_{k=1}^m l \left(\|w - w^*\|_{B_0} + \|w - w^*\|_{B_0} \right) \leq 2ml\|w - w^*\|_{B_0}.$$

Thus,

$$\|T(w) - T(w^*)\| \leq 2ml\|w - w^*\|_{B_0}. \tag{30}$$

Combining (25) and (30), we can conclude that the operator T is contraction.

Step-5: In this step we establish priori bounds.

To complete this step, it is sufficient to prove that the set

$$P = \{w \in B_0 : w = \lambda T(w/\lambda) + \lambda S(w), \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Let $w \in P$, then we have $w = \lambda T(w/\lambda) + \lambda S(w)$. Thus, for each $(x, y) \in J$, we have

$$w(x, y)$$

$$= \lambda \sum_{k=1}^{m} \left(\|I_k \frac{\left(w\left(x_k^-, y\right)\right)}{\lambda}\| - \|I_k \frac{\left(w\left(x_k^-, 0\right)\right)}{\lambda}\| \right)$$

$$+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1 - 1} (y - t)^{r_2 - 1} f(s, t, \bar{w}_{(s,t)} + v_{(s,t)}) dt ds$$

$$+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} f(s, t, \bar{w}_{(s,t)} + v_{(s,t)}) dt ds.$$

(31)

Using the assumptions (A_2) and (A_3) in (31), we have

$$\|w(x, y)\|$$

$$\leq \sum_{k=1}^{m} \lambda \left(\|I_k \frac{\left(w\left(x_k^-, y\right)\right)}{\lambda}\| - \|I_k(0)\| + \|I_k \frac{\left(w\left(x_k^-, 0\right)\right)}{\lambda}\| - \|I_k(0)\| \right)$$

$$+ 2\lambda \sum_{k=1}^{m} \|I_k(0)\| + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1}$$

$$\|\bar{w}_{(s,t)} + v_{(s,t)}\|_B dt ds + \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} dt ds$$

$$+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} \|\bar{w}_{(s,t)} + v_{(s,t)}\|_B dt ds$$

$$+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} dt ds$$

$$\leq l \sum_{k=1}^{m} \left(\|(w(t_k^-, x)\| + \|w(t_k^-, 0)\| \right) + 2I^*$$

$$+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} \|\bar{w}_{(s,t)} + v_{(s,t)}\|_B dt ds$$

$$+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} dt ds$$

$$+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} \|\bar{w}_{(s,t)} + v_{(s,t)}\|_B dt ds$$

$$+ \frac{\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_0^x \int_0^y (x - s)^{r_1 - 1} (y - t)^{r_2 - 1} dt ds,$$

where $I^* = \sum_{k=1}^{m} \|I_k(0)\|$ and

$$\|\bar{w}_{(s,t)} + v_{(s,t)}\|_B$$

$$\leq \|\bar{w}_{(s,t)}\|_B + \|v_{(s,t)}\|_B$$

$$\leq K \sup\{w(\bar{s}, \bar{t}) : (\bar{s}, \bar{t}) \in [0, s] \times [0, t]\} + K\|\phi(0,0)\| + M\|\phi\|_B$$

$$= \sigma(s, t) \, (say).$$

Hence, for each $(t, x) \in J$, we have

$$\|w(x, t)\| \leq l \sum_{k=1}^{m} \left(\|w(t_k^-, x)\| + \|w(t_k^-, 0)\| \right) + 2I^* + \frac{2a^{r_1} b^{r_2} \|q\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)}$$

$$+ \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \left[\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t - s)^{r_1 - 1} (x - \eta)^{r_2 - 1} \sigma(s, \eta) d\eta ds \right. \tag{32}$$

$$\left. + \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - \eta)^{r_2 - 1} \sigma(s, \eta) d\eta ds \right].$$

Using the inequality (32) and the definition of σ, we have

$$\sigma(t, x) \leq M\|\phi\|_B + K\|\phi(0, 0)\| + l \sum_{k=1}^{m} \left(\|w(t_k^-, x)\| + \|w(t_k^-, 0)\| \right) + 2I^*$$

$$+ \frac{2a^{r_1} b^{r_2} \|q\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} + \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \left[\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_0^x (t - s)^{r_1 - 1} \right.$$

$$\left. (x - \eta)^{r_2 - 1} \sigma(s, \eta) d\eta ds + \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - \eta)^{r_2 - 1} \sigma(s, \eta) d\eta ds \right].$$

For any $(t, x) \in J$, the Lemma 2.8 implies that there exists $\bar{\delta} = \bar{\delta}(r_1, r_2)$ such that

$$\sigma(t, x)$$

$$\leq \left(M\|\phi\|_B + K\|\phi(0, 0)\| + l \sum_{k=1}^{m} \left(\|w(t_k^-, x)\| + \|w(t_k^-, 0)\| \right) + 2I^* \right.$$

$$\left. + \frac{2a^{r_1} b^{r_2} \|q\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \right) \times \left(1 + \bar{\delta} \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t - s)^{r_1 - 1} (x - \eta)^{r_2 - 1} d\eta ds \right)$$

$$\leq \left(M\|\phi\|_B + K\|\phi(0, 0)\| + l \sum_{k=1}^{m} \left(\|w(t_k^-, x)\| + \|w(t_k^-, 0)\| \right) + 2I^* \right.$$

$$\left. + \frac{2a^{r_1} b^{r_2} \|q\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \right) \times \left(1 + \bar{\delta} \frac{a^{r_1} b^{r_2} \|p\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \right)$$

$$= \bar{\tau} \, (say).$$

Hence, for each $(t, x) \in J$, we have

$$\|w(t, x)\|_\infty \leq \sigma(t, x). \tag{33}$$

This proves that the set P is bounded.

Therefore, by the consequence of Theorem 2.9 (Burton-Kirk fixed point theorem) we deduce that $S + T$ has a fixed point which is a solution of IVP given by (1) to (4). This completes the proof. $\square$

4. Applications

In this section, we provide an illustrative example to support the analytic proof of our Theorem 3.1.

Example 4.1.

Consider the following IVP for FOIPHDEs with infinite delay:

$$\left(D_{z_k}^r u \right)(x, y) = \frac{e^{-x-y}}{9 + e^{x+y}} \cdot \frac{2 + |u(x, y)|}{1 + |u(x, y)|}, \; (x, y) \in J = \left[0, \frac{1}{2} \right] \times [0, 1] \bigcup \left(\frac{1}{2}, 1 \right] \times [0, 1], \tag{34}$$

$$u\left(\frac{1}{2}^{+}, y\right) = u\left(\frac{1}{2}^{-}, y\right) + \frac{\left|u\left(\frac{1}{2}^{-}, y\right)\right|}{\frac{1}{4} + \left|u\left(\frac{1}{2}^{-}, y\right)\right|}, \; if \; y \in [0, 1], \tag{35}$$

$$u(x, y) = x + y^2, \; if \; (x, y) \in [-1, 1] \times [-2, 1] \setminus (0, 1] \times (0, 1], \tag{36}$$

$$u(x, 0) = x, \; x \in [0, 1], \; u(0, y) = y^2, \; y \in [0, 1], \tag{37}$$

where $z_0 = (0, 0), z_1 = \left(\frac{1}{2}, 0\right)$. Let $\sigma \in \mathbf{R}$ and C_σ be the set of all piece-wise continuous functions ϕ : $(-\infty, 0] \times (-\infty, 0] \to \mathbf{R}^n$ for which a limit $\lim_{\|(s,t)\| \to \infty} e^{\sigma(s+t)} \phi(s, t)$ exists with the norm

$$\|\phi\|_{C_\sigma} = \sup_{(s,t) \in (-\infty, 0] \times (-\infty, 0]} e^{\sigma(s+t)} \|\phi(s, t)\|.$$

Set

$$f(x, y, \varphi) = \frac{e^{-x-y}(2 + |\varphi|)}{(9 + e^{x+y})(1 + |\varphi|)}, \; (x, y) \in [0, 1] \times [0, 1], \; \varphi \in C, \tag{38}$$

and

$$I_1(u) = \frac{|u|}{\frac{1}{4} + |u|}, \; u \in \mathbf{R}. \tag{39}$$

Then, it is clear that the functions f and I_1 are continuous, and for $(x, y) \in [0, 1] \times [0, 1]$ and $\varphi \in C$, we have

$$|f(x, y, \varphi)| \leq \frac{e^{-x-y}}{(9 + e^{x+y})}(2 + |\varphi|). \tag{40}$$

Hence assumption (A_2) is satisfied with

$$p(x, y) = \frac{2e^{-x-y}}{(9 + e^{x+y})} \; and \; q(x, y) = \frac{e^{-x-y}}{(9 + e^{x+y})}.$$

Now, for $u_1, u_2 \in \mathbf{R}$, we obtain

$$|I_1(u_1) - I_2(u_2)| = \left|\frac{|u_1|}{\frac{1}{4} + |u_1|} - \frac{|u_2|}{\frac{1}{4} + |u_2|}\right| \leq \frac{1}{4}|u_1 - u_2|. \tag{41}$$

Thus assumption (A_3) is satisfied with $l = \frac{1}{4}$. Since $l = \frac{1}{4}$, then the condition (25) is also satisfied. That is all the assumptions of Theorem 3.1 are satisfied. Therefore, the Theorem 3.1 implies that the IVP given by (34) to (37) has at least one solution on $(-\infty, 1] \times (-\infty, 1]$.

5. Conclusion

In this article, we have proven a new existence criterion for solutions to the initial value problem for FOIPHDEs given by (1) to (4), applying Burton-Kirk fixed point theorem. Using our Theorem 3.1, one can easily be checked the existence of at least one solution of initial value problem given by (1) to (4). The established result provides an easy and straightforward technique to cheek the existence of solutions to the initial value problem for FOIPHDEs given by (1) to (4). Furthermore, the results of this research extend the corresponding results of Abbas *et al.* [19], Benchohra and Boutefal [27], and Abbas and Benchohra [28, 29, 30, 31].

Competing Interests

The authors declare that they have no any competing interests.

Authors Contributions

All authors read and approved the final version of the manuscript.

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