THE GENERALISED CURVATURES OF A CONGRUENCE OF CURVES IN THE SUBSPACE OF A FINSLER SPACE

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The use of the process of δ-differentation (RUND [4] [1]), pp. 55) leads to the use of DUPIN's indicatrix in finding out the principal directions of a congruence of curves. The principal directions are indeterminate by using this process as has been shown by RUND [4] and ELIOPOULOS [1]. The use of the process of Δ-differentiation which we have introduced in this paper requires the use of the osculating-indicatrix corresponding to a direction x'^i in finding out the principal directions of a congruence of curves. This method gives a linear eigenvalue problem for the determination of a congruence of curves and thus the number of principal directions are determined.

1. Introduction.

The metric function of our FINSLER space F_n refered to local coordinates x^i $(i, j=1, 2, \dots, n)$ is denoted by $F(x, x')$, it being assumed that this function satisfies the conditions usually imposed on a FINSLER metric [4]. Let F_m be a FINSLER subspace with local coordinates u^α $(\alpha, \beta = 1, 2, \dots, m)$. The metric tensors $g_{\alpha\beta}(u, u')$ and $g_{ij}(x, x')$ of F_m and F_n are connected by the relation

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')\, B^i_\alpha\, B^j_\beta \tag{1.1}$$

where

$$B^i_\alpha = \frac{\partial x^i}{\partial x^\alpha} \qquad \text{and} \qquad u'^\alpha = \frac{du^\alpha}{ds}\,. \tag{1.2}$$

There exists two sets of $(n\text{-}m)$ normals in a FINSLER subspace; one $n^i_{(\mu)}$ [2] independent of x'^i and the other $n^{*i}_{(\mu)}$ depending on x'^i. The vectors $n^i_{(\mu)}$ satisfy the relations :

$$n_{(\mu)j}\, B^j_\alpha = g_{ij}(x, n_{(\mu)})\, n^i_{(\mu)}\, B^j_\alpha = 0, \tag{1.3}$$

$$n_{(\mu)j}\, n^j_{(\mu)} = g_{ij}(x, n_{(\mu)})\, n^i_{(\mu)}\, n^j_{(\mu)} = 1, \tag{1.4}$$

$$g_{ij}(x, n_{(\tau)})\, n^i_{(\tau)}\, n^j_{(\nu)} = a_{(\tau\nu)}. \tag{1.5}$$

1) Numbers in the square brackets refer to the references at the end of the paper.
2) $\mu, \nu, \tau, \dots$ vary from $m+1$ to n.

There exists $(n\text{-}m)$ symmetric tensors independent of the direction $x^{i\prime}$. These are given by the relation

$$\gamma_{(\mu)\alpha\beta} = g_{ij}(x, n_{(\mu)}) B^i_\alpha B^j_\beta \ . \tag{1.6}$$

The covariant derivative $I^i_{\alpha\beta}$ of B^i_α with respect to u^β is given by [1]

$$I^i_{\alpha\beta} = \sum_\nu B_{(\nu)\,\alpha\beta}\, n^i_{(\nu)} + W^i_{\alpha\beta} \tag{1.7}$$

where

$$W^i_{\alpha\beta}\, n_{(\nu)i} = 0 \tag{1.8}$$

and

$$I^i_{\alpha\beta}\, n_{(\nu)i} = \sum_\nu B_{(\nu)\alpha\beta}\, a_{(\tau\nu)} = \Omega_{(\nu)\alpha\beta} \ . \tag{1.9}$$

The covariant derivative of the unit normal $n^i_{(\mu)}$ is given by

$$n^i_{(\mu);\beta} = A^\delta_{(\mu)\beta} B^i_\delta + \sum_x \nu^{(x)}_{(\mu)}\, n^i_{(x)} \tag{1.10}$$

where

$$A^\delta_{(\mu)\beta} = -\gamma^{\alpha\delta}_{(\mu)}\, \Omega_{(\mu)\alpha\beta} - \gamma^{\alpha\delta}_{(\mu)}\, g_{ij;k}(x, n_{(\mu)}) B^k_\beta B^i_\alpha n^j_{(\beta)} \tag{1.11}$$

and

$$n^j_{(\mu);\beta}\, n_{(\nu)j} = \sum_x \nu^{(x)}_{(\mu)\beta}\, a_{(x\tau)} \ . \tag{1.12}$$

2. Δ-differentiation.

Let there be a vector-field X^i and a curve C: $x^i = x^i(s)$ in F_n. Then the δ-differential of X^i in the direction of x'^i is given by

$$\frac{\delta X^i}{\delta s} = X^i_{;k}(x, x')\, x'^k \tag{2.1}$$

where

$$X^i_{;k}(x, x') = \frac{\partial X^i}{\partial x^k} + P^{*i}_{hk}(x, x')\, X^h x'^k \tag{2.2}$$

is the partial δ-differential [4] of the vector X^i with respect to the metric of F_n and

$$x'^i = \frac{dx^i}{ds}$$

is the unit tangent vector to the curve C. The differential (2.1) is in the particular direction x'^i and may be called the process of directional differentiation.

The process of δ-differentiation may further be generalised. Let there be a unit vector-field $\xi^i(S)$ which determines a congruence of curves such that one curve of the congruence[1] passes through each point of F_n. The unit vector-field ξ^i at any point is tangential to the curve of the congruence through that point. Then the covariant differential of the vector-field X^i in the direction of ξ^i at a point P of F_n is given by

(2.3) $$\frac{\Delta X^i}{\Delta S} = X^i{}_{;k}(x, x')\,\xi^k$$

where $X^i{}_{;k}(x, x')$ is the partial δ-differential as usual and $\xi^i(S)$ is the unit tangent to the curve $(x^i = x^i(S))$ of the congruence at P. The process of differentiation given by (2.3) may be called the Δ-differentiation [3] of the vector X^i in the direction ξ^i at P in F_n.

The expression $\frac{\Delta X^i}{\Delta S}$ may be called the generalised covariant differential in the sense that if the unit tangent vector to C and the unit vector-field ξ^i coincide, we get the δ-differential[4].

In the following sections, by using the Δ-differentiation, we shall define the generalised curvatures of a congruence of curves in the FINSLER subspace.

3. Generalised Absolute Curvature of a Congruence of Curves.

Let $\lambda^i_{(\mu)}$ be the contravariant component of a unit vector in the direction of a curve of the congruence such that one curve of the congruence passes through each point of P_m. The congruence is of general nature. Hence the vectors with components $\lambda^i_{(\mu)}$ may be expressed as a linear combination of the tangents B^i_α and the normals $n^i_{(\mu)}$ by the relation

(3.1) $$\lambda^i_{(\mu)} = t^\alpha_{(\mu)} B^i_\alpha + \sum_\nu C_{(\mu\nu)}\, n^i_{(\nu)}$$

where the tangential and normal components $t^\alpha_{(\mu)}$ and $C_{(\mu\nu)}$ of the congruence are respectively given by

(3.2) $$t^\alpha_{(\mu)}\, \gamma_{(\mu)\alpha\beta} = g_{ij}(x, n_{(\mu)})\, \lambda^i_{(\mu)} B^j_\beta$$

and

(3.3) $$C_{(\mu\nu)} = g_{ij}(x, n)\, \lambda^i_{(\mu)}\, n^j_{(\nu)}\,.$$

The covariant differential of (3.1) with respect to u^β is given by

(3.4) $$\lambda^i_{(\mu);\beta} = t^\alpha_{(\mu);\beta} B^i_\alpha + t^\alpha_{(\mu)} I^i_{\alpha\beta} \sum_\nu C_{(\mu\nu);\beta}\, n^i_{(\nu)} + \sum_\nu C_{(\mu\nu)}\, n^i_{(\nu);\beta}\,.$$

Using the equations (1.7) and (1.10), the equation (3.4) reduces to the form

1) The curve C is not necessarily a curve of the congruence.

$$\lambda_{(\mu)\,;\,\beta}(x, x') = q^{\alpha}_{(\mu)\beta} B^{i}_{\alpha} + \sum_{\nu} v_{(\mu\nu)\,\beta}\, n^{i}_{(\nu)} + t^{\alpha}_{(\mu)}\, w^{i}_{\alpha\beta}, \tag{3.5}$$

where

$$q^{\alpha}_{(\mu)\,\beta} = t^{\alpha}_{(\mu)\,;\,\beta} + \sum_{\nu} C_{(\mu\nu)}\, A^{\alpha}_{(\mu)\,\beta} \tag{3.6}$$

and

$$v_{(\mu\nu)\,\beta} = C_{(\mu\nu)\,;\,\beta} + t^{\alpha}_{(\mu)}\, B_{(\nu)\alpha\beta} + \sum_{\nu} C_{(\mu\sigma)}\, \delta^{(\nu)}_{(\sigma)\beta}\,. \tag{3.7}$$

The Δ-differential[1] of the congruence $\lambda^{i}_{(\mu)}$ relative to the direction ξ^{i} is

$$\frac{\Delta\lambda^{i}{}_{(\mu)}}{\Delta S} = \lambda^{i}_{(\mu)\,;\,\beta}\,\xi^{\beta} = (q^{\alpha}_{(\mu)\,\beta} B^{i}_{\alpha} + \sum_{\nu} v_{(\mu\nu)\beta}\, n^{i}_{(\nu)} + t^{\alpha}_{(\mu)}\, w^{i}_{\alpha\beta})\,\xi^{\beta} \tag{3.8}$$

where q's and v's are given by (3.6) and (3.7).

Definition (3.1). The components $\dfrac{\Delta\lambda^{i}_{(\mu)}}{\Delta S}$ may be called the components of the generalised absolute curvature vector of the congruence $\lambda^{i}_{(\mu)}$ and the scalar ${}_{\lambda}K_{(\mu)}$ given by

$$\lambda\, K^{2}_{(\mu)} \overset{\text{def}}{=\!=} gij\,(x\,,\,x') \left(\frac{\Delta\lambda^{i}_{(\mu)}}{\Delta S}\right) \left(\frac{\Delta\lambda^{i}_{(\mu)}}{\Delta S}\right) \tag{3.9}$$

is called the generalised absolute curvature of the congruence $\lambda^{i}_{(\mu)}$ relative to the direction ξ^{i} along C.

If x'^{i} and ξ^{i} coincide, the generalised absolute curvature vector reduces to

$$\frac{\delta\lambda^{i}_{(\mu)}}{\delta s} = \left(q^{\alpha}_{(\mu)\beta}\, B^{i}_{\alpha} + \sum v_{(\mu\nu)}\, n^{i}_{(\nu)} + t^{\alpha}_{(\mu)}\, w^{i}_{\alpha\beta} \right) u'^{\beta}\,.$$

In the case of Riemannian space, it will reduce to

$$\frac{\delta\lambda^{i}_{(\mu)}}{\delta s} = \Big\{ (t^{\alpha}_{(\mu)\,;\,\beta} - \sum_{\nu} C_{(\mu\nu)}\, \Omega_{(\nu)}{}^{\beta\gamma} g^{\alpha\gamma})\, B^{i}_{\alpha} + \sum_{\nu} \Big(C_{(\mu\nu)\,;\,\beta} + t^{\alpha}_{(\mu)}\, \Omega_{(\nu)\,\alpha\beta} +$$

$$+ \sum_{\sigma} C_{(\mu\sigma)}\, n^{j}_{(\sigma)\,;\,\beta}\, n_{(\nu)\,j} \Big)\, n^{i}_{(\nu)} \Big\}\, u'^{\beta}$$

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1) When a parallel displacement is taken along the element of support and ξ^{i} is taken at all points of the curve C, it reduces to CARTAN's the covariant differential [4].

which is the absolute curvature vector of the congruence $\lambda^i_{(\mu)}$ defined by MISHRA and SHRI KRISHNA [2].

4. **Generalised Normal Curvature** [1] **of a Congruence of Curves.**

Definition (4.1) The scalar ${}_{\lambda}K_{(\mu\nu)}n$ defined by

$$(4.1) \qquad {}_{\lambda}K_{(\mu\nu)}\, n\,(x_1, x', \xi) \overset{\text{def}}{=} n_{(\nu)\,i} \frac{\Delta\gamma^i_{(\mu)}}{\Delta S}\frac{1}{\chi}$$

where

$$(4.2) \qquad g\,ij\,(x_1\, x')\,\xi^i\,\xi^i = g\,\alpha\beta\,(u_1\; u')\,\xi^\alpha\,\xi^\beta = \chi^2$$

may be called the generalised normal curvature of the congruence $\lambda^i_{(\mu)}$ in the direction of ξ^i.

Theorem (4.1) : The generalised normal curvature and its square arc respectively given by

$$(4.3) \qquad {}_{\lambda}K_{(\mu\nu)}\, n\;(x; x', \xi) = \frac{1}{x}\sum_{\tau} v_{(\mu\nu)\,\beta}\, a_{(\nu\tau)}\,\xi^\beta$$

and

$$(4.4) \qquad {}_{\lambda}K^2{}_{(\mu)}\, a\;(x, x', \xi) = \frac{\varphi_{(\mu)\,\alpha\beta}\,(u, u')\,\xi^\alpha\,\xi^\beta}{g_{\alpha\beta}\,(u, u')\,\xi^\alpha\,\xi^\beta}$$

where

$$(4.5) \qquad \varphi_{(\mu)\,\alpha\beta}\,(u, u') = \sum_{\nu}\Big(\sum_{\tau} v_{(\mu\tau)\,\beta}\, a_{(\nu\tau)}\Big)\Big(\sum_{\vartheta} v_{(\mu\vartheta)}\, a_{(\nu\vartheta)}\Big)\,.$$

Proof. Applying the condition (4.1) in (3.8) and using the equations (1.3), (1.5) and (1.8), we get (4.3). With the help of (4.3) and (4.5), we get (4.4).

Definition (4.2) A direction ξ^α in F_m for which the generalised normal curvature vanishes, is called the generalised asymptotic ditection of the congruence $\lambda^i_{(\mu)}$.

Definition (4.3) A curve C whose direction at each point of it is asymptotic, is called the generalised asymptotic line of the congruence $\lambda^i_{(\mu)}$.

Theorem (4.2) The generalised asymptotic line of the congruence is given by

$$(4.6) \qquad \varphi_{(\mu)\,\alpha\beta}\,\xi^\alpha\,\xi^\beta = 0\,.$$

Proof. Using the definitions (4.2) and (4.3) and the relation (4.4). we get (4.6).

Theorem (4.3) When the congruence $\lambda^i_{(\mu)}$ has no components along the normals and the tangential components $t^\alpha_{(\mu)}$ of the congruence $\lambda^i_{(\mu)}$ coincide with the unit vector-field ξ^α the generalised normal curvature and its square are given by

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1) RUND has defined the normal curvature of the hypersurface by the relation

$$n_i \frac{\delta x'^i}{\delta s} = -x'^i \frac{\delta n_i}{\delta s}\,.$$

(4.7) $$ {}_{\lambda}K_{(\mu)}\, n(u, u', \xi) = \frac{\Omega_{(\mu)\,\alpha\beta}(u, u')\, \xi^{\alpha}\xi^{\beta}}{\{g_{\alpha\beta}(u, u')\,\xi^{\alpha}\xi^{\beta}\}^{1/2}} $$

and

(4.8) $$ {}_{\lambda}K^{2}_{(\mu)}\, n(u, u', \xi) = \frac{\sum\limits_{\mu} \Omega_{(\mu)\,\alpha\beta}\, \Omega_{(\mu)\,\delta\sigma}\, \xi^{\alpha}\xi^{\beta}\xi\ \ \xi^{\sigma}}{\{g_{\alpha\beta}\,\xi^{\alpha}\xi^{\beta}\}^{1/2}\{g_{\gamma\delta}\,\xi^{\gamma}\xi^{\sigma}\}^{1/2}} \; . $$

Proof. By the conditions given in the theorem, the tangential and normal components $t^{\alpha}_{(\mu)}$ and $C_{(\mu\nu)}$ respectively reduce to

(4.9) $$ t^{\alpha}_{(\mu)} = \xi^{\alpha} $$

and

(4.10) $$ C_{(\mu\nu)} = 0 \; . $$

Using (4.9) and (4.10) in (4.3) and (4.4), the results (4.7) and (4.8) follow.

It is obvious that if x'^{i} and ξ^{i} coincide, the generalised normal curvature of the congruence given by (4.7) reduces to the normal curvature of the FINSLER subspace with respect to the curve C defined by ELIOPOULUS [1].

In a similar manner, the equation of the generalised asymptotic line of the congruence reduces to the asymptotic line of the congruence $\lambda^{i}_{(\mu)}$ with respect to the curve C.

Definition (4.4) With respect to the normal $n^{i}_{(\mu)}$ at P of F_n and corresponding to x'^{i}, a direction ξ^{i} for which the generalised normal curvature ${}_{\lambda}K^{2}_{(\mu)}\,n$ of the congruence assumes an extreme value, is called a generalised principal direction of the congruence $\lambda^{i}_{(\mu)}$.

Theorem (4.4) The generalised principal directions of the congruence $\lambda^{i}_{(\mu)}$ are given by

(4.11) $$ ({}_{\lambda}K^{2}_{(\nu\gamma)}\, n\, g_{\alpha\beta} - \varphi_{(\nu)\,\alpha\beta})\, \xi^{\beta}_{(\gamma)} = 0 \quad (\nu = 1, 2, \ldots, m) \; . $$

Proof. To find the extreme values of ${}_{\lambda}K^{2}_{(\nu)}\,n$ for the principal directions, we have to seek the solutions of the equation

(4.12) $$ \frac{\partial}{\partial \xi^{\gamma}} \Big[\, {}_{\lambda}K^{2}_{(\nu)}\, n\, g_{\alpha\beta}\, \xi^{\alpha}\xi^{\beta} - \varphi_{(\nu)\,\alpha\beta}\, \xi^{\alpha}\xi^{\beta} \,\Big] = 0 \; . $$

Simplifying (4.12), we have

(4.13) $$ ({}_{\lambda}K^{2}_{(\nu)}\, n\, g_{\alpha\beta}(u, u') - \varphi_{(\nu)\,\sigma\beta}(u, u')\, \xi^{\beta} = 0 \; . $$

There are m linear equations in $\xi^{1}, \xi^{2}, \ldots, \xi^{m}$ not all these components being zero. Thus with respect to each normal $n^{i}_{(\nu)}$ and corresponding to any fixed direction u'^{α} of the subspace, there exists m roots of the equation

(4.14) $$ | K^{2}_{(\nu)}\, n\, g_{\alpha\beta}(u, u') - \varphi(u, u') | = 0 $$

These roots will be called the generalised principal normal curvatures of the congruence $\lambda^i_{(\mu)}$ corresponding to x'^i. We thereby obtain m generalised principal directions given by

$$(4.15) \qquad (\lambda K^2{}_{(\nu\gamma)}\, n g_{\alpha\beta} - \varphi_{(\nu)\,\alpha\beta})\,\xi^\beta{}_{(\nu)} = 0 \quad (\nu = 1,2,\ldots,m)$$

Theorem (4.5) At a point P of the subspace. with respect to the normal $n^i_{(\nu)}$, and corresponding to an arbitrary fixed direction u'^α, any two of the generalised principal directions are orthogonal and satisfy the relation

$$(4.16) \qquad \varphi_{(\nu)\,\alpha\beta}(u,u')\,\xi^\alpha{}_{(\gamma)}\,\xi^\beta{}_{(\delta)} = 0$$

Proof. Let the equations (4.13) have simple roots and let $\xi^\alpha_{(\nu)}$ and $\xi^\alpha_{(\gamma)}$ be any two of the m generalised principal directions ξ^α of the congruence, we can write

$$(4.17) \qquad (\lambda K^2{}_{(\nu\gamma)}\, n g_{\alpha\beta} - \varphi_{(\nu)\,\alpha\beta})\,\xi^\alpha{}_{(\gamma)} = 0$$

$$(4.18) \qquad (\lambda K^2{}_{(\nu\delta)}\, n g_{\alpha\beta} - \varphi_{(\nu)\,\alpha\beta})\,\xi^\alpha{}_{(\delta)} = 0$$

multiplying (4.17) by $\xi^\beta_{(\delta)}$ and (4.18) by $\xi^\beta_{(\gamma)}$ and subtracting we obtain

$$(4.19) \qquad g_{\alpha\beta}(n,u')\,\xi^\alpha_{(\gamma)}\,\xi^\beta_{(\delta)} = 0$$

which gives the condition of orthogonality. Also we have the equation (4.16).

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ÖZET

Bir eğri kongrüansının esas doğrultularının belirtilmesi problemi, "δ-türev" usulünün (RUND [4], s. 55) kullanılması halinde DUPIN göstergesinin kullanılmasını icabettirir. Bu usulün esas doğrultuları belirtemiyeceği gerek RUND [4] gerek ELIOPOULOS [1] tarafından gösterilmiştir. Bu yazıda tanımlanan "Δ-türev" bir eğri kongrüansının esas doğrultularının bulunması için bir x'^i doğrultusunda öskülâtör göstergelerin kullanılmasını gerektirmektedir. Bu yoldan hareket edildiğinde bir eğri kongrüansının esas doğrultuları için bir lineer eigen-değer problemine varılır ve esas doğrultuların sayısı elde edilir.