ON A CLASS OF INTEGRAL FUNCTIONS DEFINED BY A DIRICHLET SERIES

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A set of m integral functions, each having a Dirichlet series representation of the form

$$f_k(s) = \sum_{n=1}^{n} a_{k,n}\, e^{s\lambda_{k,n}}$$

where the $\lambda_{k,n}$ form a monotonic increasing sequence satisfying certain special conditions at the limits, is considered and the function given by the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} A_{m,n}\, e^{s\mu_{m,n}}$$

with

$$A_{m,n} \sim \prod_{k=1}^{m} (a_{k,n})^{p_k};\ \mu_{m,n} \sim \sum_{k=1}^{m} p_k \lambda_{k,n};\ p_k > 0$$

is defined and some properties of this function are obtained as consequence of those of the function $f_k(s)$.

1. Consider the Dirichlet series

$$F(s) = \sum_{n=1} a_n\, e^{s\lambda_n}$$

where

$$\{\lambda_n\} \uparrow \infty\,,\quad \lambda_1 \geq 0\,,\quad s = \sigma + it$$

and

(1.1) $$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0.$$

Let σ_c and σ_a be the abscissa of convergence and abscissa of absolute convergence, respectively, of $F(s)$. If $\sigma_c = \sigma_a = \infty$, $F(s)$ is an integral function. We shall suppose throughout that (1.1) holds and that $\sigma_c = \sigma_a = \infty$.

Let $f(s)$, $k = 1, 2, \ldots, m$, be m integral functions of $s = \sigma + it$ with the following Dirichlet series representation,

$$f_k(s) = \sum_{n=1} a_{k,n}\, e^{s\lambda_{k,n}}, \tag{1.2}$$

where

$$\{\lambda_{k,n}\} \uparrow \infty, \qquad \lambda_{k,1} \geq 0,$$

in the whole s-plane and let

$$f(s) = \sum_{n=1} A_{m,n}\, e^{s\mu_{m,n}}, \tag{1.3}$$

where

$$A_{m,n} \sim \prod_{k=1}^{m} (a_{k,n})^{p_k}, \quad \mu_{m,n} \sim \sum_{k=1}^{m} p_k\, \lambda_{k,n}$$

and the p 's are positive constants.

In this paper, we have obtained some of the properties of $f(s)$.

2. **Theorem 1**

If $f_k(s)$, as defined in (1.1), *are m integral functions of order*

$$\varrho_k \ (0 \leq \varrho_k \leq \infty), \qquad (k = 1, 2, \ldots, m),$$

such that

(*i*)
$$\frac{\lambda_{k,n}}{\lambda_{1,n}} \to \frac{\alpha_k}{\alpha_1}$$

as

$$n \to \infty \qquad (\alpha_k > 0)$$

and

(*ii*)
$$\lim_{n\to\infty} (\lambda_{k,n+1} - \lambda_{k,n}) = h,$$

where $h\,(\leq 1)$ is a positive constant, then the function $f(s)$, defined by (1.3), *is also an integral function of order ϱ, such that*

$$\frac{1}{\varrho} \geq \sum^{m} \frac{p_k}{\varrho_k}.$$

Proof.

It is known ([1], p. 80) that $f(s)$, as defined by (1.3), is an integral function.

It is also well known ([2], p. 44) that if

$$F(s) = \sum_{n=1}^{\infty} a_n\, e^{s\lambda_n}, \quad \{\lambda_n\} \uparrow \infty, \quad \lambda_1 \geq 0$$

and

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = h,$$

where $h\,(\leq 1)$ is a positive constant, then $F(s)$ is of linear order ϱ, (2.1) $$\varrho = \limsup_{n\to\infty} \frac{\log \lambda_n}{\log \left| \dfrac{a_n}{a_{n+1}} \right|}.$$

The corresponding result for $f_k(s)$ and $f(s)$ are

$$\frac{1}{\varrho_k} = \liminf_{n\to\infty} \frac{\log \left| \dfrac{a_{k,n}}{a_{k,n+1}} \right|}{\log \lambda_{k,n}},$$

and

$$\frac{1}{\varrho} = \liminf_{n\to\infty} \frac{\log \left| \dfrac{A_{m,n}}{A_{m,n+1}} \right|}{\log \mu_{m,n}}.$$

Hence

$$\frac{1}{\varrho} = \liminf_{n\to\infty} \frac{\log \prod_{k=1}^{m} \left| \dfrac{a_{k,n}}{a_{k,n+1}} \right|^{p_k}}{\log \left(\sum_{k=1}^{m} p_k \lambda_{k,n} \right)}$$

or

$$\frac{1}{\varrho} = \liminf_{n\to\infty} \frac{p_k \text{ iog} \prod_{k=1}^{m} \left| \dfrac{a_{k,n}}{a_{k,n+1}} \right|}{\log \lambda_{k,n}},$$

since

$$\log \left(\sum_{k=1}^{m} p_k \lambda_{k,n} \right) \sim \log \left(\frac{L \lambda_{k,n}}{\alpha_k} \right),$$

where $L = p_1 \alpha_1 + p_2 \alpha_2 + \cdots + p_m \alpha_m$, so

$$\log \left(\sum_{k=1}^{m} p_k \lambda_{k,n} \right) \sim \log \lambda_{k,n}$$

and therefore

$$\frac{1}{\varrho} \geq \sum_{k=1}^{m} \left\{ \liminf_{n\to\infty} \frac{p_k \log \left| \dfrac{a_{k,n}}{a_{k,n+1}} \right|}{\log \lambda_{k,n}} \right\} = \sum_{k=1}^{m} \frac{p_k}{\varrho_k}.$$

3. Theorem 2

If $f_k(s)$, as defined in (1.2), are m integral functions of lower order ν_k $(0 \leq \nu_k \leq \infty)$, $(k = 1, 2, \ldots, m)$ such that

(*i*) $$\frac{\lambda_{k,n}}{\lambda_{1,n}} \to \frac{\alpha_k}{\alpha_1} \quad as \quad n \to \infty \quad (\alpha_k > 0)$$

and

(*ii*) $$\lim_{n\to\infty} (\lambda_{k,n+1} - \lambda_{k,n}) = h,$$

where h (≤ 1) is a positive constant, and

(*iii*)
$$\frac{\log\left|\frac{a_{k,n}}{a_{k,n+1}}\right|}{\lambda_{k,n+1}-\lambda_{k,n}},$$

are non-decreasing functions of n for $n > n_0$, then the function $f(s)$, as defined in (1.3), *is also an integral function of lower order ν such that*

$$\frac{1}{\nu} \leq \sum_{k=1}^{m} \frac{p_k}{\nu_k}.$$

Proof.

We know that $f_i(s)$ is an integral function, ([1], p. 80), also we have ([2], p. 44) that if $F(s)$ represented by the DIRICHLET series $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ in the whole plane is an integral function, such that

(*i*)
$$\lim_{n\to\infty} (\lambda_{n+1}-\lambda_n) = h,$$

where $h\,(\leq 1)$ is a positive constant, and

(*ii*)
$$\frac{\log\left|\frac{a_n}{a_{n+1}}\right|}{\lambda_{n+1}-\lambda_n},$$

is a non-decreasing function of n for $n > n_0$, then the lower linear order ν of $F(s)$ is given by

(3.1)
$$\nu = \lim_{n\to\infty} \inf \frac{\log \lambda_n}{\log\left|\frac{a_n}{a_{n+1}}\right|}.$$

Using (3.1) in the case of $f_k(s)$, we have

$$\frac{1}{\nu_k} = \lim_{n\to\infty} \sup \frac{\log\left|\frac{a_{k,n}}{a_{k,n+1}}\right|}{\log \lambda_{k,n}}.$$

Futhermore

$$\frac{1}{\nu} = \lim_{n\to\infty} \sup \frac{\log\left|\frac{A_{m,n}}{A_{m,n+1}}\right|}{\log \mu_{m,n}} = \lim_{n\to\infty} \sup \frac{\log \prod_{k=1}^{m} \left|\frac{a_{k,n}}{a_{k,n+1}}\right|^{p_k}}{\log\left(\sum_{k=1}^{m} p_k \lambda_{k,n}\right)}$$

$$= \lim_{n\to\infty} \sup \frac{\log \prod_{k=1}^{m} \left|\frac{a_{k,n}}{a_{k,n+1}}\right|^{p_k}}{\log \lambda_{k,n}}.$$

Hence

$$\frac{1}{\nu} \leq \sum_{k=1}^{m} p_k \left\{ \limsup_{n\to\infty} \frac{\log \left| \dfrac{a_{k,n}}{a_{k,n+1}} \right|}{\log \lambda_{k,n}} \right\} = \sum_{k=1}^{m} \frac{p_k}{\nu_k} .$$

The following corailaries easily follow from the above theorems :

Corollary 1

If $f(s)$ is an integral function of infinite order then each of the functions $f_k(s)$ is of infinite order.

Corollary 2

If any of the integral functions $f_k(s)$ is of zero order then the order of $f(s)$ is zero.

Corollary 3

If the functions $f_k(s)$ satisfying the conditions of theorem 2, are of linearly regular growth, then $f(s)$ is also of regular growth, and

$$\frac{1}{\varrho} = \sum_{k=1}^{m} \frac{p_k}{\varrho_k} .$$

Since for functions of linearly regular growth $\varrho = \nu$, we can write,

$$\frac{1}{\nu} \leq \frac{p_1}{\nu_1} + \frac{p_2}{\nu_2} + \cdots + \frac{p_m}{\nu_m} = \frac{p_1}{\varrho_1} + \frac{p_2}{\varrho_2} + \cdots + \frac{p_m}{\varrho_m} .$$

But $\varrho \geq \nu$, therefore,

$$\frac{1}{\nu} = \frac{p_1}{\varrho_1} + \frac{p_2}{\varrho_2} + \cdots + \frac{p_m}{\varrho_m} = \frac{1}{\varrho} .$$

Hence $f(s)$ is also of linearly regular growth.

4. **Theorem 3.**

If $f_k(s)$, $(k = 1, 2, \ldots, m)$, as defined in (1.2), be m integral functions of order ϱ_k $(0 < \varrho_k < \infty)$ and type T_k $(0 < T_k < \infty)$ such that

(*i*) $$\limsup_{n\to\infty} \frac{\log n}{\lambda_{k,n}} = 0,$$

and

(*ii*) $$\frac{\lambda_{k,n}}{\lambda_{1,n}} \to \frac{\alpha_k}{\alpha_1} \quad as \quad n \to \infty \quad (\alpha_k > 0),$$

then the function $f(s)$, *as defined in* (1.3), *is also an integral function of order* ϱ *and type* T, *such that*

$$\left(\frac{\varrho T}{L}\right)^{\frac{1}{\varrho}} \leq \prod_{k=1}^{m} \left(\frac{\varrho_k T_k}{\alpha_k}\right)^{\frac{p_k \alpha_k}{L \rho_k}},$$

provided

$$\frac{L}{\varrho} = \sum_{\nu=1}^{m} \frac{p_\nu \alpha_\nu}{\varrho_\nu}$$

where

$$L = \sum_{\nu=1}^{m} p_\nu \alpha_\nu .$$

Proof.

Let

$$R(n) = (\mu_{m,n})^{\frac{1}{\rho}} \mid A_{m,n}\mid^{\frac{1}{\mu_{m,n}}} \sim \left(\frac{L \lambda_{1,n}}{\alpha_1}\right)^{\frac{1}{\rho}} \left| \prod_{k=1}^{m} a_{k,n}^{p_k} \right|^{\frac{\alpha_1}{L \lambda_{1,n}}}$$

$$= L^{\frac{1}{\rho}} \left| \prod_{k=1}^{m} \frac{\lambda_{k,n}^{\frac{1}{\rho_k}} a_{k,n}^{\frac{1}{\lambda_{k,n}}}}{\alpha^{\frac{1}{\rho_k}}} \right|^{\frac{p_k \alpha_k}{L}} .$$

Now let

$$R_k(n) = \lambda_{k,n}^{\frac{1}{\rho_k}} \mid a_{k,n} \mid^{\frac{1}{\lambda_{k,n}}},$$

then

$$\limsup_{n\to\infty} R_k(n) = (e \varrho_k T_k)^{\frac{1}{\rho_k}} .$$

Futhermore

$$(e \varrho T)^{\frac{1}{\rho}} = \limsup_{n\to\infty} R(n) \leq L^{\frac{1}{\rho}} \prod_{k=1}^{m} \limsup_{u\to\infty} \left| \frac{\lambda_{k,n}^{\frac{1}{\rho_k}} a_{k,n}^{\frac{1}{\lambda_{k,n}}}}{\alpha_k^{\frac{1}{\rho_k}}} \right|^{\frac{p_k \alpha_k}{L}}$$

$$= L^{\frac{1}{\rho}} \prod_{k=1}^{m} \left(\frac{e \varrho_k T_k}{\alpha_k}\right)^{\frac{p_k \alpha_k}{L \rho_k}} .$$

Hence

$$\left(\frac{\varrho\, T}{L}\right)^{\frac{1}{\rho}} \leq \prod^{m} \left(\frac{\varrho_k\, T_k}{\alpha_k}\right)^{\frac{p_k\, \alpha_k}{L\, \rho_k}} .$$

5. Theorem 4

If $f_k(s)$ are m integral functions of order ϱ_k $(0 < \varrho_k < \infty)$, $(k = 1, 2, \ldots, m)$ and lower type t_k $(0 < t_k < \infty)$, such that

(*i*)
$$\lim_{n \to \infty} \sup \frac{\log n}{\lambda_{k,n}} = 0,$$

(*ii*)
$$\frac{\lambda_{k,n}}{\lambda_{1,n}} \to \frac{\alpha_k}{\alpha_1} \quad as \quad n \to \infty,$$

and

(*iii*)
$$\frac{\log \left|\dfrac{a_{k,n}}{a_{k,n+1}}\right|}{\lambda_{k,n+1} - \lambda_{k,n}},$$

forms a non-decreasing function of n for $n > n_0$, then the function $f(s)$ is also an integral function of order ϱ and lower type t, such that

$$\left(\frac{\varrho\, t}{L}\right)^{\frac{1}{\rho}} \geq \prod_{k=1}^{m} \left(\frac{t_k\, \varrho_k}{\alpha_k}\right)^{\frac{p_k\, \alpha_k}{L\, \rho_k}},$$

provided

$$\frac{L}{\varrho} = \sum_{\nu=1}^{m} \frac{p_\nu\, \alpha_\nu}{\varrho_\nu},$$

where

$$L = \sum_{\nu=1}^{m} p_\nu\, \alpha_\nu .$$

Proof.

Let

$$R(n) = (\mu_{m,n})^{\frac{1}{\rho}}\, |A_{m,n}|^{\frac{1}{\mu_{m,n}}} \sim L^{\frac{1}{\rho}} \left| \prod_{k=1}^{m} \frac{\lambda_{k,n}^{\frac{1}{\rho_k}}\, a_{k,n}^{\frac{1}{\lambda_{k,n}}}}{\alpha_k^{\frac{1}{\rho_k}}} \right|^{\frac{p_k\, \alpha_k}{L}},$$

as in theorem 3.

Now let

$$R_k(n) = (\lambda_{k,n})^{\frac{1}{\rho_k}} \, |a_{k,n}|^{\frac{1}{\lambda_{k,n}}},$$

then

$$\lim_{n\to\infty} \inf R_k(n) = (e \, \varrho_k \, t_k)^{\frac{1}{\rho_k}} .$$

Hence

$$(e \, \varrho \, t)^{\frac{1}{\rho}} = \lim_{n\to\infty} \inf R(n) \geq L^{\frac{1}{\rho}} \prod_{k=1}^{m} \lim_{n\to\infty} \inf \left| \frac{\lambda_{k,n}^{\frac{1}{\rho_k}} \, a_{k,n}^{\frac{1}{\lambda_{k,n}}}}{\alpha_k^{\frac{1}{\rho_k}}} \right|^{\frac{p_k \alpha_k}{L}}$$

$$= L^{\frac{1}{\rho}} \prod_{k=1}^{m} \left(\frac{e \, \varrho_k \, t_k}{\alpha_k} \right)^{\frac{p_k \alpha_k}{L \rho_k}} .$$

Therefore,

$$\left(\frac{\varrho \, t}{L} \right)^{\frac{1}{\rho}} \geq \prod_{k=1}^{m} \left(\frac{\varrho_k \, t_k}{\alpha_k} \right)^{\frac{p_k \alpha_k}{L \rho_k}} . \quad (1)$$

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ÖZET

$\{\lambda_{k,n}\}$ dizileri monoton artan ve limitleri bazı özel şartlar sağlamak üzere,

$$f_k(s) = \sum_{n=1}^{\infty} a_{k,n}\, e^{s\lambda_{k,n}}$$

DIRICHLET serileri ile verilen m tane fonksiyon göz önüne alınmak ve

$$A_{m,n} \sim \prod_{k=1}^{m} (a_{k,n})^{p_k}\,; \quad \mu_{m,n} \sim \sum_{k=1}^{m} p_k\, \lambda_{k,n}\,; \quad p_k > 0$$

olmak üzere

$$f(s) = \sum_{n=1}^{\infty} A_{m,n}\, e^{s\mu_{m,n}}$$

DIRICHLET serisiyle tanımlanan $f(s)$ fonksiyonu göz önüne alınmaktadır. Bu fonksiyonun bazı özellikleri, $f_k(s)$ fonksiyonlarının özelliklerinin sonuçları olarak elde edilmektedir.