MAGNETO-HYDRODYNAMIC FLOW OF A VISCOELASTIC FLUID BETWEEN TWO CONDUCTING POROUS PLATES

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The present paper deals with the flow of a conducting viscoelastic fluid between two porous plates under a transverse magnetic field with constant suction and injection, and finite electrical wall conductivities. Using the perturbation method, an exact solution is obtained for small relaxation time. It is found that velocity decreases with the increase in the viscoelastic parameter, whereas the induced magnetic field increases with the increase in both the plate conductivity and the viscoelastic parameter. Also various other conclusions are drawn.

0. Notation :

C'_0, C'_1, C_{10}, C_{20}, C_{30}, C_{40}, C_{11}, C_{21}, C_{31}, C_{41} Constants

e_{ij} Strain rate tensor

H_0 Applied transverse magnetic field

h Dimensionless induced axial magnetic field

L Half-width between plates

M HARTMANN number

m Suction parameter

p' Hydrostatic pressure

P Dimensionless hydrostatic pressure

$\bar{p}_{ij}$ Deviatoric stress tensor

p_{ij} Dimensionless deviatoric stress tensor

p_{rm} Magnetic PRANDTL number

R REYNOLDS number

R_m Magnetic REYNOLDS number

u Dimensionless axial velocity

$\bar{x}$ Longitudinal distance

x Dimensionless longitudinal distance

$\bar{y}$ Transverse distance

y Dimensionless longitudinal distance

ε Viscoelastic parameter

λ Relaxation time

μ Viscosity

ν Kinematic viscosity

μ_e Magnetic permeability

η Electric diffusivity

φ_l, φ_u Electrical conductivity parameters of lower and upper plates

ϱ Density

σ Electrical conductivity

δ_{ij} KRONECKER tensor

π_{ij} Stress tensor

1. **Introduction:** The flow of a conducting newtonian fluid between two parallel plates under transverse magnetic field was first studied by HARTMANN and LAZARUS [1]. GUPTA [2] has studied POISEUILLE flow with suction and injection. Later KAPUR and RATHY [3] investigated the same problem for a conducting viscoelastic fluid, taking the walls to be nonconducting. But in the flow process, the percolation of the fluid through the plates makes them electrically conducting, even though the plates themselves are non conducting when they are in dry state. Thus electrical conductivity of the plates plays a significant part. Hence we cannot neglect the conductivities of the plates, when we are dealing with magneto-hydrodynamic flow problems with suction and injection.

The aim of the present paper is to study the effect of wall conductivity in this problem. In this paper we have studied the steady incompressible flow of a viscoelastic fluid between two conducting parallel plates under a transverse magnetic field with constant suction and injection.

In the analysis a rectangular Cartesian co-ordinate system is used.

$$\vec{H}(H_x, H_0, 0),\ \vec{V}(u_x, v_0, 0),\ \vec{E}(0, 0, E_z)\ \vec{J}(0, 0, J_z)$$

denote the magnetic, velocity, electric and current field vectors respectively. It is assumed that all the flow variables depend on $\bar{y}$ only.

2. **Basic Equations.** The Rheological equation is

$$\pi_{ij} = -p'\delta_{ij} + \bar{p}_{ij}\,, \tag{2.1}$$

$$\bar{p}_{ij} + \lambda\widetilde{\bar{p}}_{ij} = 2\mu e_{ij}\,, \tag{2.2}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

$$\widetilde{\bar{p}}_{ij} = \frac{\partial}{\partial t}\bar{p}_{ij} + \bar{p}_{ij,R}\, v_R - \bar{p}_{iR}\, v_{j,R} - \bar{p}_{Rj}\, v_{i,R} + \bar{p}_{ij}\, v_{R,R}\,.$$

With the above assumptions from the basic equations of magneto-fluid-dynamics and the rheological equations, we get

$$0 = -\frac{\partial p'}{\partial \bar{y}} - \frac{\mu_e H_x}{4\pi}\frac{\partial H_x}{\partial \bar{y}} - \frac{\partial \bar{p}_{yy}}{\partial \bar{y}}\,, \tag{2.3}$$

(2.4) $$\varrho v_0 \frac{\partial u_x}{\partial \bar{y}} = -\frac{\partial p'}{\partial \bar{x}} + \frac{\partial \bar{p}_{xy}}{\partial \bar{y}} + \frac{\mu_e H_0}{4\pi} \frac{\partial H_x}{\partial \bar{y}} ,$$

(2.5) $$\eta \frac{\partial^2 H_x}{\partial \bar{y}^2} + H_0 \frac{\partial u_x}{\partial \bar{y}} - v_0 \frac{\partial H_x}{\partial \bar{y}} = 0 ,$$

(2.6) $$\bar{p}_{yy} + \lambda v_0 \frac{\partial \bar{p}_{yy}}{\partial \bar{y}} = 0 ,$$

(2.7) $$\bar{p}_{xy} + \lambda \left(v_0 \frac{\partial \bar{p}_{xy}}{\partial \bar{y}} - \frac{\partial u_x}{\partial \bar{y}} \bar{p}_{yy} \right) = \mu \frac{\partial u_x}{\partial \bar{y}} ,$$

(2.8) $$\bar{p}_{xx} + \lambda \left(v_0 \frac{\partial \bar{p}_{xx}}{\partial \bar{y}} - 2 \frac{\partial u_x}{\partial \bar{y}} \bar{p}_{xy} \right) = 0 ,$$

(2.9) $$\bar{p}_{xx} = \bar{p}_{xz} = \bar{p}_{yz} = 0 .$$

Using the transformations,

$$\bar{x} = Lx , \bar{y} = Ly , v_0 = m u_0 , u_x = u_0 u , H_x = H_0 h ,$$

$$\bar{p}' = p u_0^2 p , \bar{p}_{xy} = \varrho u_0^2 p_{xy} , \bar{p}_{yy} = \varrho u_0^2 p_{yy} , \bar{p}_{xx} = \varrho u_0^2 p_{xx} ,$$

(2.10) $$\varepsilon = \frac{\lambda u_0}{L}, \ R = \frac{\varrho u_0 L}{u} , R_m = 4\pi \mu_e u_0 L , M = \mu_e H_0 L \left(\frac{\sigma}{u} \right)^{1/2},$$

we get the non-dimensional form of equations (2.4) to (2.7) as

(2.11) $$m \frac{du}{dy} = -\frac{\partial p}{\partial x} + \frac{d}{dy} p_{xy} + \frac{M^2}{RR_m} \frac{dh}{dy} ,$$

(2.12) $$p_{rm} \frac{d^2 h}{dy^2} + R \frac{du}{dy} - mR \frac{dh}{dy} = 0 ,$$

(2.13) $$p_{yy} + \varepsilon m \frac{d}{dy} p_{yy} = 0 ,$$

(2.14) $$p_{xy} + \varepsilon \left(m \frac{d}{dy} p_{xy} - p_{yy} \frac{du}{dy} \right) = \frac{1}{R} \frac{du}{dy} .$$

It can be shown that $\frac{\partial p}{\partial x}$ is constant. Hence we take $\frac{\partial p}{\partial x} = -A$, where A is a positive constant.

Boundary Conditions: Let the two walls be given by $y = \pm 1$. From the no slip condition, we get

(2.15) $$u = 0 \quad \text{at} \quad y = \pm 1 .$$

The condition that the tangential component of $\vec{E}$ and $\vec{H}$ must be continuous across the interface ([4], [5]), gives the following boundary condition for the induced magnetic field

$$\frac{dh}{dy} - \frac{1}{\varphi_l} h = 0 \qquad \text{at} \quad y = -1,$$

(2.16)

$$\frac{dh}{dy} + \frac{1}{\varphi_u} h = 0 \qquad \text{at} \quad y = +1.$$

3. **Solution**: Integration of (2.11) gives

(3.1) $$mu = Ay + p_{xy} + \frac{M^2}{RR_m} h + C_1^1 .$$

Eliminating p_{yy} between (2.12) and (2.13), we get

$$\varepsilon^2 \left(\frac{du}{dy} \frac{d^2 p_{xy}}{dy^2} - \frac{d^2 u}{dy^2} \frac{dp_{xy}}{dy} \right) + \varepsilon \left(2 \frac{du}{dy} \frac{dp_{xy}}{dy} - \frac{d^2 u}{dy^2} p_{xy} \right)$$

(3.2) $$+ \left(p_{xy} - \frac{1}{R} \frac{du}{dy} \right) \frac{du}{dy} = 0.$$

We develop the solution in powers of $\varepsilon (\varepsilon << 1)$, hence we lake

$$u = \sum_0^\infty \varepsilon^n u_n \; , \; h = \sum_0^\infty \varepsilon^n h_n \; , \; C_1^1 = \sum_0^\infty \varepsilon^n C_{1\,(n)}^1 \; ,$$

(3.3) $$p_{xy} = \sum_0^\infty \varepsilon^n p_{(n)\,xy} \, , p_{yy} = \sum_0^\infty \varepsilon^n p_{(n)\,yy} \, , p_{xx} = \sum_0^\infty \varepsilon^n p_{(n)\,xx} \, .$$

Substituting (3.3) in (2.12), (3.1) and (3.2) and equating the various powers of ε, we shall get the equations for determining

$$u_n , h_n \text{ and } p_{(n)xy} \text{ for } n = 0, 1,2, \ldots$$

The boundary conditions will give

(3.4) $$u_n = 0 \quad (n = 0, 1, 2, \ldots) \quad \text{at} \quad y = \pm 1 ,$$

(3.5) $$\frac{dh_n}{dy} - \frac{1}{\varphi_l} h_n = 0, (n = 0,1,2 \ldots) \text{ at } y = -1$$

and

$$\frac{dh_n}{dy} + \frac{1}{\varphi_u} h_n = 0, (n = 0,1,2,\ldots) \text{ at } y = +1.$$

Zero Order Solution. Substituting (3.3) in (2.12), (3.1) and (3.2) and taking zero order terms in ε , we have

(3.6) $$p_{rm} \frac{d^2 h_0}{dy^2} + R \frac{du_0}{dy} - m R \frac{dh_0}{dy} = 0,$$

(3.7) $$mu_0 = Ay + p_{(0)\,xy} + \frac{M^2}{RR_m} h_0 + C'_{1(0)} ,$$

(3.8) $$p_{(0)\,xy} = \frac{1}{R}\frac{du_0}{dy} .$$

Eliminating u_0 and $p_{(0)\,xy}$ from (3.6), (3.7) and (3.8), we have

(3.9) $$\frac{d^2h_0}{dy^2} - mR\left(1+\frac{1}{p_{rm}}\right)\frac{dh_0}{dy} + \frac{R}{R_m p_{rm}}(m^2 RR_m - M^2)h_0 = \frac{AR^2}{p_{rm}} y + C'_0 .$$

Solving this, we get

(3.10) $$h_0 = C_{10} e^{\alpha y} + C_{20} e^{\beta y} - \frac{ARR_m}{M^2 - m^2 RR_m} y + C_{80} .$$

From equations (3.6) and (3.10), we get

(3.11) $$h_0 = C_{10}\left(m - \frac{p_{rm}\alpha}{R}\right)e^{\alpha y} + C_{20}\left(m - \frac{p_{rm}\beta}{R}\right)e^{\beta y} + \frac{mARR_m}{M^2 - m^2 RR_m} y + C_{80}$$

where α, β are zeroes of

(3.12) $$\lambda^2 - mR\left(1+\frac{1}{p_{rm}}\right)\lambda + \frac{R}{R_m p_{rm}}(m^2 RR_m - M^2) = 0 .$$

Using the boundary conditions (3.4), (3.5), we get

$$u_0 = \frac{ARRm}{M^2 - m^2 RR_m}\left[\left\{\frac{(Q_4 Q_2 sh\,\beta - mQ_6)\ \ Q_1(e^{\alpha y} - ch\alpha) + (mQ_5 - Q_4 Q_1 sh\,\alpha)\,Q_2(e^{\beta y} - ch\,\beta)}{(Q_6 Q_2 sh\,\beta - Q_1 Q_6 sh\,\alpha)}\right\} - m^y\right],$$

(3.13)

$$h_0 = \frac{ARR_m}{M^2 - m^2 RR_m}\left[\left\{\frac{(Q_4 Q_2 sh\beta - mQ_6)[c^{\alpha y} - (\alpha\varphi_4 + 1)e^{\alpha}] + (mQ_5 - Q_4 Q_1 sh\alpha)\,[e^{\beta y} - (\beta\varphi_u + 1)e^{\beta}]}{(Q_5 Q_2 sh\,\beta - Q_1 Q_6 sh\,\alpha)}\right\}, - \left\{y - (1+\varphi_u)\right\}\right]$$

(3.14)

where

$$Q_1 = \left(m - \frac{p_{rm}\alpha}{R}\right),\ Q_2 = \left(m - \frac{p_{rm}\beta}{R}\right),\ Q_4 = 2 + \varphi_l + \varphi_u ,$$

$$Q_5 = \alpha(\varphi_u e^{\alpha} + \varphi_l e^{-\alpha}) + 2\,sh\,\alpha ,\ Q_6 = \beta(\varphi_u e^{\beta} + \varphi_l e^{-\beta}) + 2sh\,\beta .$$

Knowing u_0, h_0 we can get all other field quantities, like, E_0, $p_{(0)xy}$, $p_{(0)yy}$, etc.

First Order Solution. Substituting (3.3) in (3.2) and taking first order terms in ε from equation (3.2), and simplifying we have

(3.15) $$p_{(1)\,xy} = \frac{1}{R}\frac{du_1}{dy} - \frac{1}{R}\frac{d^2u_0}{dy^2} .$$

Proceeding in a similar way as in the zero order solution, we get the equation determining h in first order as

$$\frac{d^2h_1}{dy^2} - mR\left(1+\frac{1}{p_{rm}}\right)\frac{dh_1}{dy} + \frac{R}{R_m p_{rm}}(m^2RR_m - M^2)h_1 \tag{3.16}$$

$$= C_1^1 - \frac{R}{p_{rm}}\left\{ C_{10}\alpha^2\left(m - \frac{p_{rm}\alpha}{R}\right)e^{\alpha y} + C_{20}\beta^2\left(m - \frac{p_{rm}\beta}{R}\right)e^{\beta y}\right\}.$$

The solution of (3.16) is

$$h_1 = C_{11}e^{\alpha y} + C_{21}e^{\beta y} - \frac{R}{p_{rm}(\alpha-\beta)}\left[C_{10}\alpha^2\left(m - \frac{p_{rm}\alpha}{R}\right)e^{\alpha y}\right. \tag{3.17}$$

$$\left. - C_{20}\beta^2\left(m - \frac{p_{rm}\beta}{R}\right)e^{\beta y}\right] y + C_3 .$$

Taking first order terms in ε from (2.12) and using (3.17), we get

$$u_1 = C_{11}\left(m - \frac{p_{rm}\alpha}{R}\right)e^{\alpha y} + C_{21}\left(m - \frac{R_{rm}\alpha}{R}\right)e^{\beta y} \tag{3.18}$$

$$+ \frac{1}{\alpha-\beta}\left[C_{10}\alpha^2\left(m - \frac{R_{rm}\alpha}{R}\right)e^{\alpha y}\left(1+\alpha y - \frac{mR}{p_{rm}}y\right)\right.$$

$$\left. - C_{20}\beta^2\left(m - \frac{p_{rm}\beta}{R}\right)e^{\beta y}\left(1+\beta y - \frac{mR}{p_{rm}}y\right)\right] + C_{41} .$$

Applying the boundary conditions (3.4), (3.5), we have

$$u_1 = C_{11}Q_1(e^{\alpha y} - e^{-\alpha}) + C_{21}Q_2(e^{\beta y} - e^{-\beta}) \tag{3.19}$$

$$+ \frac{1}{\alpha-\beta}\left[C_{10}\alpha^2 Q_1\left\{ e^{\alpha y}\left(1+\alpha y - \frac{mR}{p_{rm}}y\right) - e^{-\alpha}\left(1-\alpha+\frac{mR}{p_{rm}}\right)\right\}\right.$$

$$\left. - C_{20}\beta^2 Q_2\left\{ e^{\beta y}\left(1+\beta y - \frac{mR}{p_{rm}}y\right) - e^{-\beta}\left(1-\beta+\frac{mR}{p_{rm}}\right)\right\}\right],$$

$$h_1 = C_{11}\left\{ e^{\alpha y} - (\alpha\varphi_u + 1)e^{\alpha}\right\} + C_{21}\left\{ e^{\beta y} - (\beta\varphi_u + 1)e^{\beta}\right\} \tag{3.20}$$

$$- \frac{R}{p_{rm}(\alpha-\beta)}\left[C_{10}\alpha^2\left(m - \frac{p_{rm}\alpha}{R}\right)\left\{ ye^{\alpha y} - \left[(1+\alpha)\varphi_u + 1\right]e^{\alpha}\right\}\right.$$

$$\left. - C_{20}\beta^2\left(m - \frac{p_{rm}\beta}{R}\right)\left\{ ye^{\beta y} - \left[(1+\beta)\varphi_u + 1\right]e^{\beta}\right\}\right]$$

where

$$Z_1 = -\frac{1}{\alpha-\beta}\left[C_{10}\alpha^2 Q_1\left\{ sh\alpha + \left(\alpha - \frac{mR}{p_{rm}}\right)ch\,\alpha\right\}\right.$$

$$\left. - C_{20}\beta^2 Q_2\left\{ sh\,\beta + \left(\beta - \frac{mR}{p_{rm}}\right)ch\,\beta\right\}\right],$$

$$Z_2 = \frac{R}{p_{rm}(\alpha-\beta)}\left[C_{10}\alpha^2 Q_1\left\{ 2ch\alpha + \left(\varphi_u e^{\alpha} + \varphi_l e^{-\alpha}\right) + \alpha\left(\varphi_u e^{\alpha} - \varphi_l e^{-\alpha}\right)\right\}\right.$$

$$\left. - C_{20}\beta^2 Q_2\left\{ 2ch\beta + \left(\varphi_u e^{\beta} + \varphi_l e^{-\beta}\right) + \beta\left(\varphi_u e^{\beta} - \varphi_l e^{-\beta}\right)\right\}\right],$$

$$Z_3=\alpha(\varphi_u e^{\alpha}+\varphi_l e^{-\alpha})+2\,sh\,\alpha \quad , \quad Z_4=\beta(\varphi_u e^{\beta}+\varphi_l e^{-\beta})+2\,sh\,\beta \ ,$$

$$Z_5=\left(m-\frac{p_{rm}\alpha}{R}\right)sh\,\alpha\ , \qquad Z_6=\left(m-\frac{p_{rm}\beta}{R}\right)sh\,\beta,$$

$$C_{11}=\frac{Z_2Z_6-Z_1Z_4}{Z_3Z_6-Z_4Z_5}\ , \qquad C_{21}=\frac{Z_1Z_3-Z_2Z_5}{Z_3Z_6-Z_4Z_5}.$$

Knowing u_1, h_1 we can evaluate E_1, $p_{(1)xy}$, $p_{(1)xx}$, $p_{(1)yy}$, etc.

n-th Order Solution: Taking n-th order terms also into account in equation (3.2) and substituting the value of $p_{(n)xy}$ in (3.1) and then eliminating, We have

$$(3.21) \qquad \frac{d^2h_n}{dy^2}-mR\left(1+\frac{1}{p_{rm}}\right)\frac{dh_n}{dy}+\frac{R}{R_m p_{rm}}(m^2RR_m-M^2)h_n=f_n(u_{n-1}, u_{n-2}, \ldots).$$

Knowing the solutions up to the $(n-1)$ th order, we know f. Then we can solve (3.21) to know h_n. Similar procedure as in zeroth order and first order solution gives u_n. Hence in this way theoretically, the solution can be found up to any order we like. But second and higher order solutions will give unwieldy expressions and the solution up to the first order gives fairly approximate values.

4. **Conclusion:** The graphs (Fig. 4.1, Fig. 4.2) showing the variation of $u(=u_0+\varepsilon u_1)$,

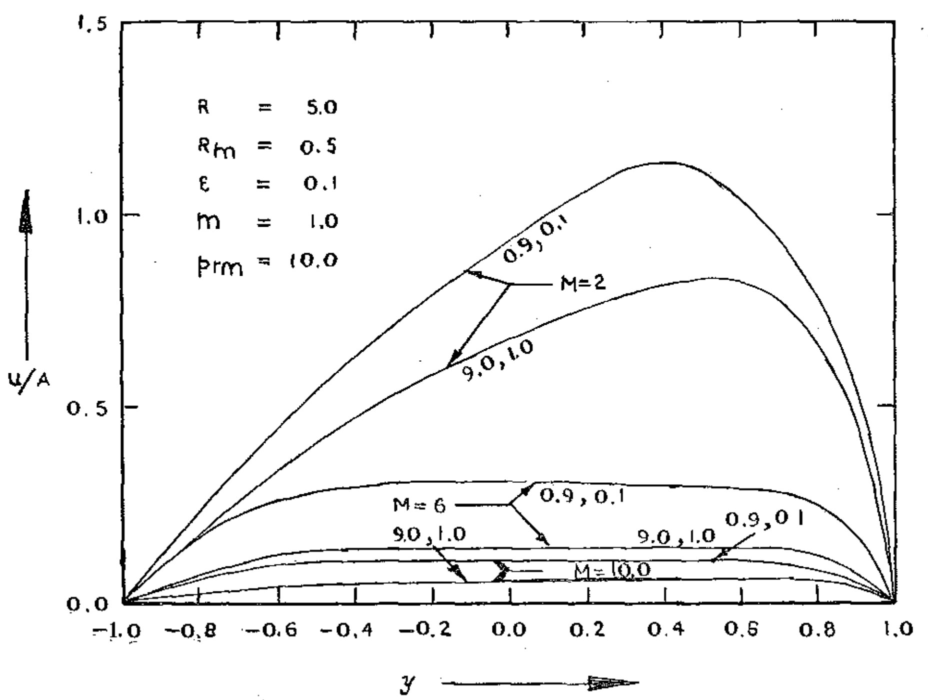


Fig. 4.1

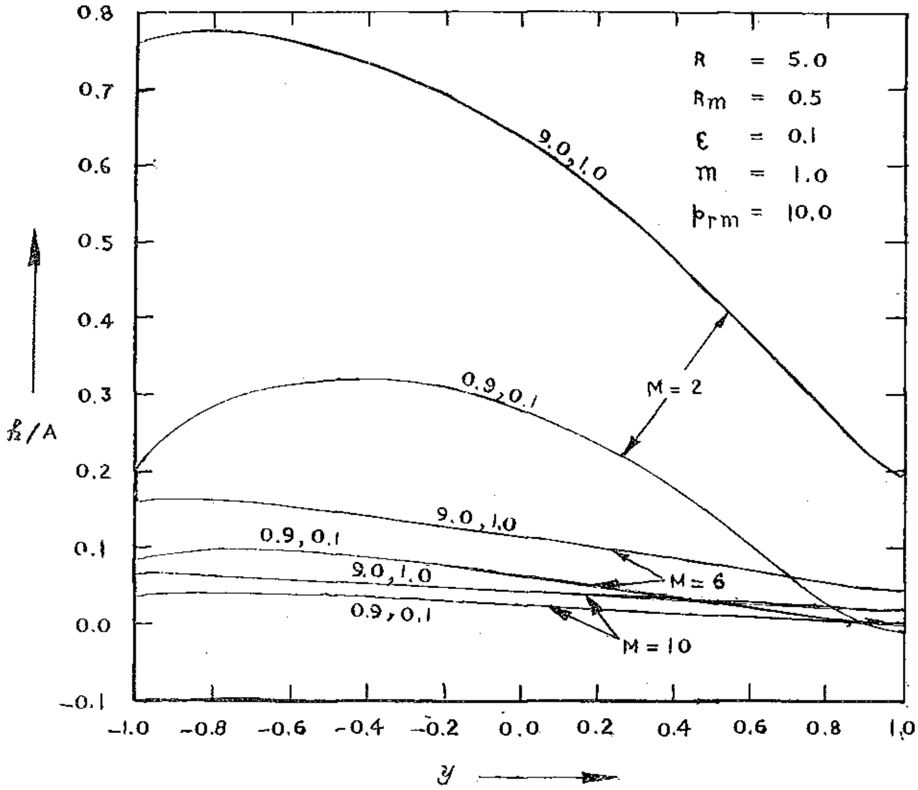


Fig. 2

and $h(=h_0+\varepsilon h_1)$ versus y are drawn for fixed values of $m, R, R_m, p_{rm}, \varepsilon$ and for various values of M and φ_l, φ_u and it is found that

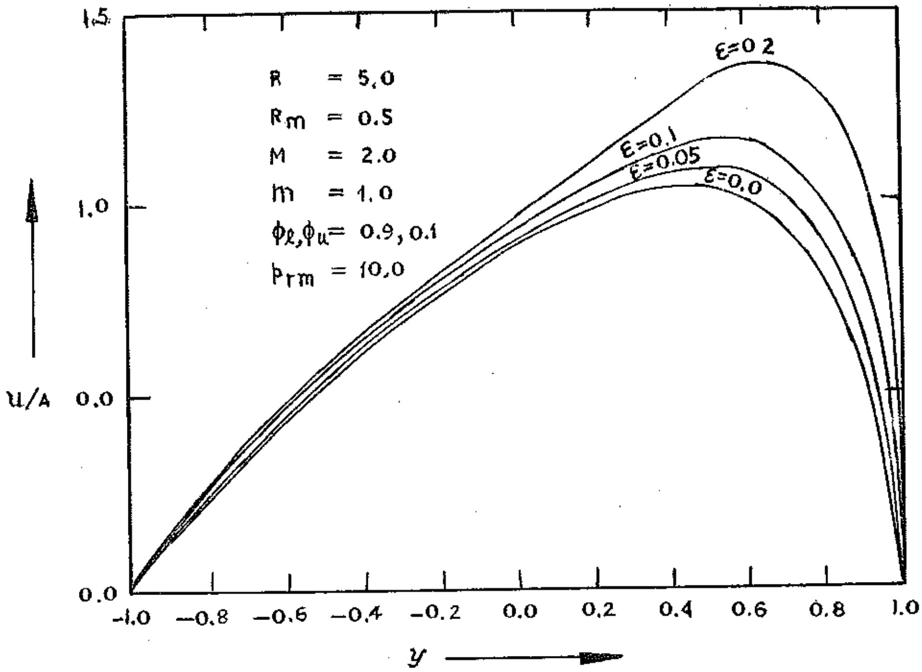
I) As the HARTMANN number increases, velocity is decreasing at every point, hence the fluid is being retarded by the increase in the magnetic field.

II) The velocity decreases with the increase in the plate conductivities.

III) Maximum of velocity moves towards the plate with suction with increase of HARTMANN number as well as with the increase in plate conductivities φ_l, φ_u.

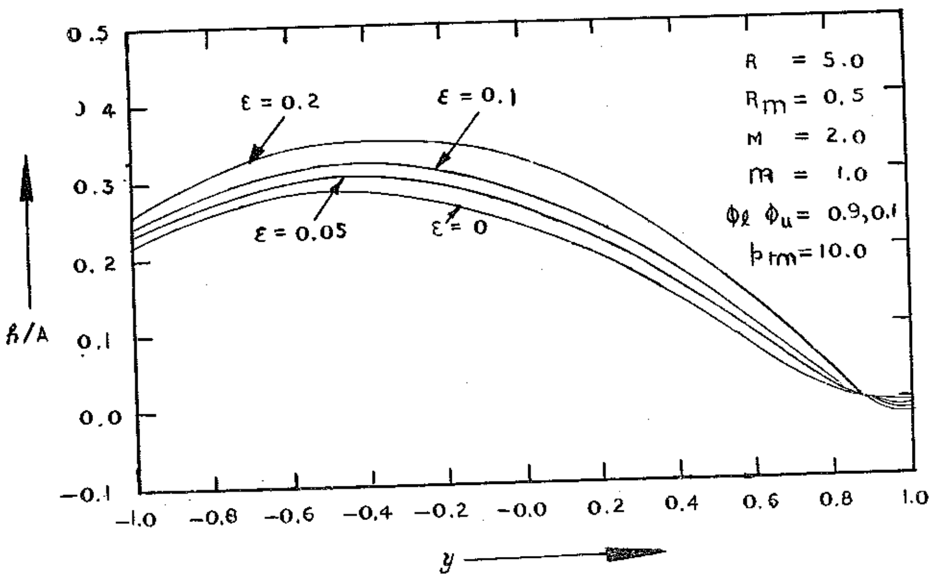
IV) Induced magnetic field increases with the increase in the plate conductivities φ_l, φ_u and decreases with the increase of HARTMANN number.

V) Induced magnetic field is more near the plate with injection, than that near the plate with suction.



R = 5.0
Rm = 0.5
M = 2.0
m = 1.0
φℓ,φu = 0.9, 0.1
prm = 10.0
ε=0.2
ε=0.1
ε=0.05
ε=0.0
u/A
y

Fig. 4.3



R = 5.0
Rm = 0.5
M = 2.0
m = 1.0
φℓ φu = 0.9,0.1
prm = 10.0
ε = 0.2
ε = 0.1
ε = 0.05
ε = 0
h/A
y

Fig. 4.4

(Fig. 4.3, Fig. 4.4) are the graphs showing the variation oi u and h versus y for fixed values of M, m, R, R_m, p_{rm} and φ_l, φ_u or various values of ε, from which it is concluded that

VI) In POISEUILLE flow the velocity increases at every point due to viscoelastic effects and the maximum of the velocity profile shifts towards the plate with suction.

VII) The induced magnetic field also increase due to viscoelastic effects. (*)

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ÖZET

Bu yazıda iletken ve viskoelastik bir akışkanın iki mesameli levha arasındaki akışı incelenmektedir : bu olayı incelerken dikine bir magnetik alanın varlığı, emiş veya verişin sabit olduğu ve duvarların elektrik iletkenliğin sonlu bulunduğu kabul edilmiştir. Pertürbasyon metodunu kullanmak suretiyle kısa bir relaksasyon zamanına tekabül eden tam çözümler elde edilmiştir. Viskoelastik parametrenin büyümesi halinde hızın azaldığını, halbuki doğurulao magnetik alanın gerek levha iletkenliği, gerek viskoelastik parametre ile birlikte büyüdüğü tesbit edilmiştir. Ayrıca çözümden daha başka sonuçlar da çıkarılmıştır.

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