ON THE MEAN VALUES OF INTEGRAL FUNCTIONS *

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Let $\mu_\delta(r)$ and $m_{\delta,k}(r)$ be the two functions associated to an integral function $f(z)$ by the formulae (1.1) and (1.2), δ and k denoting any two positive numbers. Three theorems concerning these two functions are proved.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an integral function of order ϱ. Also, let

$$\mu_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\vartheta})|^\delta \, d\vartheta, \tag{1.1}$$

$$m_{\delta,k}(r) = \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} |f(x e^{i\vartheta})|^\delta x^k \, dx \, d\vartheta, \tag{1.2}$$

where δ and k are any positive numbers.

We shall obtain some of the properties of $\mu_\delta(r)$ and $m_{\delta,k}(r)$.

2. **Theorem 1.** *Let $f(z)$ be an integral function. Then, for $0 < r_1 < r_2$,*

$$2(r_2^{k+1} - r_1^{k+1})\, \mu_\delta(r_1) \leq (k+1) \{ r_2^{k+1} m_{\delta,k}(r_2) - r_1^{k+1} m_{\delta,k}(r_1) \} \leq 2(r_2^{k+1} - r_1^{k+1})\, \mu_\delta(r_2)$$

where δ and k are any positive numbers.

Proof. From (1.1) and (1.2), we have

$$m_{\delta,k}(r) = \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} |f(x e^{i\vartheta})|^\delta x^k \, dx \, d\vartheta$$

$$= \frac{2}{r^{k+1}} \int_0^r \mu_\delta(x)\, x^k \, dx \, d\theta .$$

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Therefore,

(2.1) $$\frac{r^{k+1}}{2} m_{\delta,k}(r) = \int_0^r \mu_\delta(x)\, x^k\, dx\, d\theta\,.$$

From (2.1) follows

(2.2) $$r_2^{k+1} m_{\delta,k}(r_2) - r_1^{k+1} m_{\delta,k}(r_1) = 2 \int_{r_1}^{r_2} \mu_\delta(x)\, x^k\, dx$$

and the inequalities follow since $\mu_\delta(x)$ is an increasing function of x.

We may note that if $f(z)$ is an integral function, other than a constant, and $\alpha\,(0<\alpha<1)$ is a constant,

$$\lim_{r\to\infty} \left\{ \frac{1}{m_{\delta,k}(r) - \alpha^{k+1} m_{\delta,k}(\alpha r)} \right\} = 0\,.$$

3. **Theorem** 2. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an integral function of order* $\varrho\,(0 < \varrho < \infty)$, *type* τ *ond lower type* ν, *then*

(*i*) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_\delta(r)}{r^\varrho} = \frac{\delta\tau}{\delta\nu}\,,$$

(*ii*) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}(r)}{r^\varrho} = \frac{\delta\tau}{\delta\nu}\,,$$

where δ *and* k *are any positive numbers,*

Proof. (*i*) We have

(3.1) $$\mu_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\vartheta})|^\delta\, d\vartheta \leq \{ M(r) \}^\delta\,,$$

where $M(r) = \max_{|z|=r} |f(z)|$.

Also, we have

(3.2) $$\mu_\delta(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\vartheta})|^\delta\, d\vartheta$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} \{ |Q_{N(r)}|\, r^{N(r)} \}^\delta\, d\vartheta$$

$$= \{ \mu(r) \}^\delta\,,$$

where $\mu(r) = |Q_{N(r)}|\, r^{N(r)}$ is the maximum term of rank $N(r)$ for $|z| = r$, in the series for $f(z)$.

From (3.1) and (3.2), we get

(3.3) $$\{\mu(r)\}^{\delta} \leq \mu_{\delta}(r) \leq \{M(r)\}^{\delta}.$$

Since for functions of finite order $\log \mu(r) \sim \log M(r)$ it follows, from (3.3),

(3.4) $$\log \{\mu_{\delta}(r)\}^{1/\delta} \sim \log M(r).$$

The result (*i*) follows easily from (3.4) since

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^{\varrho}} = \begin{matrix}\tau\\ \nu\end{matrix}.$$

(*ii*) From (2.1), we have

$$m_{\delta,k}(r) \leq \frac{2}{r^{k+1}} \mu_{\delta}(r) \int_0^r x^k \, dx$$

$$= \frac{2}{(k+1)} \mu_{\delta}(r),$$

since $\mu_{\delta}(x)$ is an increasing function of x.

Taking limits, we get

(3.5) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}(r)}{r^{\varrho}} \leq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_{\delta}(r)}{r^{\varrho}} = \begin{matrix}\delta\tau\\ \delta\nu\end{matrix}.$$

Also, from (2.1), we have for $a > 0$

$$m_{\delta,k}\{r(1+a)\} = \frac{2}{\{r(1+a)\}^{k+1}} \int_0^{r(1+a)} \mu_{\delta}(x)\, x^k \, dx$$

$$\geq \frac{2}{r^{k+1}(1+a)^{k+1}} \int_r^{r(1+a)} \mu_{\delta}(x)\, x^k \, dx$$

$$\geq \frac{2\mu_{\delta}(r)}{(k+1)} \{1-(1+a)^{-k-1}\},$$

since $\mu_{\delta}(x)$ is an increasing function of x.

Taking limits, we get

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}\{r(1+a)\}}{\{r(1+a)\}^{\varrho}} \geq \frac{1}{(1+a)^{\varrho}} \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_{\delta}(r)}{r^{\varrho}}$$

or,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}(r)}{r^{\varrho}} \geq \frac{1}{(1+a)^{\varrho}} \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_{\delta}(r)}{r^{\varrho}}.$$

Since the left hand side is independent of a, for $a \to 0$, we get

(3.6) $$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{\delta,k}(r)}{r^{\varrho}} \geq \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu_{\delta}(r)}{r^{\varrho}} = \begin{matrix}\delta\tau\\ \delta\nu\end{matrix}.$$

The result (*ii*) follows from (3.5) and (3.6).

4. **Theorem 3.** *Let $f(z)$ be an integral function. Then*

$$\lim_{r\to\infty} \sup \frac{m_{\delta,k}(r)}{\{M(r)\}^{\varrho}} \leq \lim_{r\to\infty} \sup \frac{m_{\delta,k}(r)}{\mu_{\delta}(r)} \leq \frac{2}{(k+1)},$$

where $M(r) = \max |f(z)|$ *and* δ,k *are any positive numbers.*

Proof. Since $\mu_{\delta}(x)$ is an increasing function of x, therefore from (2.1), we have

$$m_{\delta,k}(r) \leq \frac{2}{r^{k+1}} \mu_{\delta}(r) \int_0^r x^k \, dx$$

$$= \frac{2}{(k+1)} \mu_{\delta}(r).$$

Taking limits, we get

(4.1) $$\lim_{r\to\infty} \sup \frac{m_{\delta,k}(r)}{\mu_{\delta}(r)} \leq \frac{2}{(k+1)}.$$

Also, from (1.1), we have

(4.2) $$\mu_{\delta}(r) \leq \{M(r)\}^{\delta}.$$

Therefore, from (4.1) and (4.2), follows

$$\lim_{r\to\infty} \sup \frac{m_{\delta,k}(r)}{\{M(r)\}^{\varrho}} \leq \lim_{r\to\infty} \sup \frac{m_{\delta,k}(r)}{\mu_{\delta}(r)} \leq \frac{2}{(k+1)}.$$

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ÖZET

$\mu_{\delta}(r)$ ve $m_{\delta,k}(r)$, $f(z)$ integral fonksiyonuna (1.1) ve (1.2) formülleri ile tekabül ettirilen iki fonksiyonu göstersin : δ ve k'nın birer pozitif sayı oldukları farz edilerek bu iki fonksiyonu ilgilendiren üç teorem ispat ediliyor.