A THEOREM ON n—DIMENSIONAL LAPLACE TRANSFORMATIONS

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A theorem on n dimensional Laplace transformations is proved and then applied to the evaluation of certain definite integrals, not easy to tackle otherwise in a neat form.

1. The present paper deals with the operational calculus based on n-dimensional Laplace transformations,

$$f(p_1, \dots, p_n) = p_1 \dots p_n \int_0^\infty \dots \int_0^\infty \exp\ (- \Sigma\, p_i\, x_i)\ \ h(x_1, \dots, x_n)\ \ dx_1 \dots dx_n$$

where h is the «original» and f, its «image»; f being the operational representation of h, expressed symbolically as $f \underset{n}{\subset} h$ or $h \underset{n}{\supset} f$.

In this paper we shall derive a theorem in n-dimensional Laplace transformations and apply this theorem (especially for $n = 2$) to evaluate certain definite integrals, not easy to tackle otherwise in a neat form. In this theorem we shall take $D(\sigma_{10}, \dots, \sigma_{n0})$ to represent the set of points for which $R(p_i) = R(p_{i0}) = \sigma_{i0} > 0$.

In establishing the theorem it has been frequently necessary to effect changes in the orders of integrations in the multiple integrals. For the justification of the process, we use Fubini's theorem and the Lemma given below.

2. **Lemma** (i). If one of the repeated integrals

$$\int_a^\infty dx \int_b^\infty |f(x, y)|\ dy \text{ and } \int_b^\infty dy \int_a^\infty |f(x, y)|\ dx$$

be finite, then the integrals of $f(x, y)$ over the domain $(a, b\, ;\ \infty, \infty)$ is finite and is equal to the repeated integrals over the same domain.

3. **Lemma** (ii). If $f(x, y)$ be non-negative and measurable and be defined in $(a, b\, ;\ \infty, \infty)$, then

$$\int_a^\infty dx \int_b^\infty f(x, y)\ dy,\ \int_b^\infty dy \int_a^\infty f(x, y)\ dx,\ \int_a^\infty \int_b^\infty f(x, y)\ dx\ dy,$$

are all finite and equal or else all infinite.

The domain of convergence of the n-dimensional LAPLACE transformation has been defined by me already in my Ph. D. thesis, submitted to B. I. T. S., (Pilani), in 1965.

4. **Theorem.** Let

$$f(p_1, \dots, p_n) \underset{n}{\subset} h(x_1, \dots, x_n) \tag{4.1}$$

$$\text{for} \quad (p_1, \dots, p_n) \in D(\sigma_{10}, \dots, \sigma_{n0});$$

and let

$$F(x) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{8x}[\Sigma(a_i x_i)]^2\right) D_{2v}\left(\frac{1}{\sqrt{2x}} \Sigma a_i x_i\right) h(x_1, \dots, x_n)\, dx_1 \dots dx_n \tag{4.2}$$

exist as an absolutely convergent integral for $R(x_i) > 0$. The a_i's arc all positive constants.

If the operational variable p corresponds to x, then

$$p^{v+\frac{1}{2}-\frac{n}{2}} f(a_1\sqrt{p}, \dots, a_n\sqrt{p}) \subset \frac{(a_1, \dots, a_n)}{2^v\sqrt{\pi}} x^{-v-\frac{1}{2}} F(x) \tag{4.3}$$

provided that $x^{-v-\frac{1}{2}} F(x)$ is integrable L in $(0,\infty)$ with respect to the variable x and $R(p) > 0$.

Proof.

Since the definition integral in (4.1) is absolutely convergent for $(p_1, \dots, p_n) \in D(\sigma_{10}, \dots, \sigma_{n0})$, it is absolutly convergent for all p_i, p_j, such that $R(p_i) = R(p_j) = R(p) > 0$. Thus the definition integral in (4.1) can be written as

$$p^{v-\frac{n+1}{2}} f(a_1\sqrt{p}, \dots, a_n\sqrt{p}) \tag{4.4}$$

$$= (a_1 \dots a_n) \int_0^\infty \dots \int_0^\infty p^{v-\frac{1}{2}} \exp[-(\Sigma a_i u_i) p^{\frac{1}{2}}]\, h(u_1, \dots, u_n)\, du_1 \dots du_n .$$

Since $p^{v-\frac{1}{2}} e^{-\sqrt{ap}} = \frac{1}{2^v\sqrt{\pi}} \int_0^\infty e^{-px} x^{-v-\frac{1}{2}} e^{-\frac{a}{8x}} D_{2v}\left(\sqrt{\frac{a}{8x}}\right) dx \quad \text{for } R(a) > 0,$

we can write (4.4) in the form

$$p^{v-\frac{n+1}{2}} f(a_1\sqrt{p}, \dots, a_n\sqrt{p}) \tag{4.5}$$

$$= \frac{(a_1 \dots a_n)}{2^v\sqrt{\pi}} \int_0^\infty \dots \int_0^\infty h(u_1, \dots, u_n)\, du_1 \dots du_n$$

$$\times \int_0^\infty e^{-px} x^{-v-\frac{1}{2}} \exp\left[-\frac{1}{8x}(\Sigma a_i u_i)^2\right] D_{2v}\left(\frac{1}{\sqrt{2x}} \Sigma a_i u_i\right) dx .$$

On changing the order of integration on the right of (4.5), by FUBINI's theorem, we obtain the required theorem.

5. Applications.

1) Let us take $f(p, q) = \dfrac{qp^{1-v}}{(1+pq)^{v+1}}$

so that

$$h(x, y) = \frac{X^v J_{2v}(2\sqrt{xy})}{\Gamma(v+1)}, \qquad R(v) > -\tfrac{1}{2}.$$

Hence from the theorem, we get

$$\frac{ab\; x^{-v-\frac{1}{2}}}{2^v\sqrt{\pi}\;\Gamma(v+1)} \int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) u^v J_{2v}(2\sqrt{uv'})\, du\, dv'$$

$$\supset \frac{a^{-2v}\, b^{-v}\, p^{\frac{v+1}{2}}}{\left(p+\dfrac{1}{ab}\right)^{v+1}}$$

or,

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) u^v J_{2v}(2\sqrt{uv'})\, du\, dv'$$

$$= \frac{\Gamma\left(\frac{v+1}{2}\right)\sqrt{\pi}\; 2^v\; x^{\frac{3v}{2}+1}}{a^{2v+1}\; b^{v+1}}\; e^{-\frac{x}{ab}}\; L_{\frac{v-1}{2}}^{\frac{v+1}{2}}\left(-\frac{x}{ab}\right), \quad R(v) > -\tfrac{1}{2}.$$

In particular, when $v = 1$, then we obtain

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_2\left(\frac{au+bv'}{\sqrt{2x}}\right) u\, J_2(2\sqrt{uv'})\, du\, dv' = \frac{2\sqrt{\pi}}{a^3 b^2}\, x^{5/2}\, e^{-\frac{x}{ab}}.$$

2) Let $f(p, q) = \dfrac{\sqrt{pq}}{p+\sqrt{2pq}+q}$; so that $h(x, y) = \dfrac{1}{\pi}\, \dfrac{\sqrt{-y+\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$.

Hence by the theorem, we get

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) \frac{\sqrt{-v'+\sqrt{u^2+v'^2}}}{\sqrt{u^2+v'^2}}\, du\, dv'$$

$$= \frac{2^v\, \pi\, x^{3/4}}{\sqrt{a}\,(a+\sqrt{2ab}+b)\,\Gamma(\frac{5}{4}-v)}, \quad R(v) < \tfrac{5}{4}.$$

3) Let $f(p, q) = \dfrac{q}{p+\sqrt{pq}}$; so that $h(x, y) = \dfrac{2}{\pi}\left[\sqrt{\dfrac{x}{y}} - \tan^{-1}\left(\sqrt{\dfrac{x}{y}}\right)\right]$.

Hence by the theorem, we get

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{9x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) \left[\sqrt{\frac{u}{v'}} - \tan^{-1}\sqrt{\left(\frac{u}{v'}\right)}\right] du\, dv'$$

$$= \frac{2^{v-1}\, \pi^{3/2}\, x}{a\,(a+\sqrt{ab})\,\Gamma(3/2-v)}, \quad R(3-2v) > 0.$$

4) Let $f(p, q) = \frac{p^2 q^2}{p^2 q^2 + 1}$; so that $h(x, y) = \operatorname{ber}(2\sqrt{xy})$.

Hence by the theorem, we get

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) \operatorname{ber}(2\sqrt{uv'})\, du\, dv' = \frac{2^v \sqrt{\pi}\, x}{ab\, \Gamma(^{3/2}-v)}$$

$$\times\ {}_1F_2\left(1; \frac{3}{4}-\frac{v}{2}, \frac{5}{4}-\frac{v}{2}; -\frac{x^2}{4a^2 b^2}\right), \quad R(3-2v) > 0.$$

5) Let $f(p, q) = \frac{1}{\sqrt{pq}} e^{-\frac{1}{pq}}$; so that $h(x, y) = (xy)^{\frac{1}{6}} J_{\frac{1}{2}, \frac{1}{2}}(3\sqrt[3]{xy})$. Then

$$\frac{(ab)^{3/2} x^{-v-\frac{1}{2}}}{2^v \sqrt{\pi}} \int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) (uv')^{\frac{1}{6}}$$

$$\times\ J_{\frac{1}{2}, \frac{1}{2}}(3\sqrt[3]{uv'})\, du\, dv' \supset p^{v-1} e^{-\frac{1}{abp}}.$$

In particular, when $v = \frac{3}{2}$, we obtain

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')}{8x}\right] D_3\left(\frac{au+bv'}{\sqrt{2x}}\right) (uv')^{\frac{1}{6}} J_{\frac{1}{2}, \frac{1}{2}}(3\sqrt[3]{uv'})\, du\, dv'$$

$$= \frac{(2x)^{3/2}}{(ab)^{3/2}} \cos\left(\frac{2\sqrt{x}}{\sqrt{ab}}\right).$$

6) Let $f(p, q) = \frac{pq}{p^3 q + 1}$; so that $h(x, y) = \frac{x^2}{2}\ {}_0F_3\left(1, 4/3, 5/3; -\frac{x^3 y}{27}\right)$.

Hence by the theorem, we obtain

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) u^2\ {}_0F_3\left(1, \frac{4}{3}, \frac{5}{3}; -\frac{u^3 v'}{27}\right) du\, dv'$$

$$= \frac{2^{v+1}\sqrt{\pi}\, x^2}{a^3 b\, \Gamma(\frac{5}{2}-v)}\ {}_1F_2\left(1; \frac{5}{4}-\frac{v}{2}, \frac{7}{4}-\frac{v}{2}; -\frac{x^2}{4a^3 b}\right).$$

7) Let $f(p, q) = \frac{\sqrt{p}\, q^{\frac{1}{2}-v}}{1+pq}$; so that $h(x, y) = x^{-v/2} y^{v-/2} H_v(2\sqrt{xy})$.

Hence by the theorem, we obtain

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{2v}\left(\frac{au+bv'}{\sqrt{2x}}\right) u^{-\frac{v}{2}} v'^{\frac{v}{2}} M_v(2\sqrt{uv'})\, du\, dv'$$

$$= \frac{\sqrt{\pi}\, 2^v x^{\frac{v+3}{2}}}{a^{3/2} b^{v+3/2} \Gamma(2-\frac{v}{2})}\ {}_1F_1\left(1; 2-\frac{v}{2}; -\frac{x}{ab}\right).$$

8) Let $f(p, q) = \frac{\Gamma(\frac{1}{2}+m)\,\Gamma(\frac{1}{2}-m)}{4\pi\sqrt{\pi}}\, p^{1-2m}\, q^{1+2m}\, e^{\frac{p^2 q^2}{32}}\, K_m\left(\frac{p^2 q^2}{32}\right),$

$\left(-\frac{1}{2} < R(m) < \frac{1}{2}\right)$; so that $h(x, y) = \left(\frac{x}{y}\right)^{2m} J_{m,-m}\,(3\,\sqrt[3]{x^2 y^2})$.

Therefore, we get (when $v = \frac{1}{2} - 2m$)

$$\int_0^\infty \int_0^\infty \exp\left[-\frac{(au+bv')^2}{8x}\right] D_{1-4m}\left(\frac{au+bv'}{\sqrt{2x}}\right)\left(\frac{u}{v'}\right)^{2m} J_{m,-m}\,(3\,\sqrt[3]{u^2 v'^2})\,du\,dv'$$

$$= \frac{\Gamma(\frac{1}{2}+m)\;\Gamma(\frac{1}{2}-m)\;b^{3m}\;e^{-\frac{2x^2}{a^2b^2}}}{\sqrt{\pi}\;2^{3m+\frac{1}{2}}\;a^m\,x^m\;\Gamma(2m+1)}\;M_{-\frac{3m}{2},\,\frac{m}{2}}\left(\frac{4x^2}{a^2b^2}\right),$$

$$-\tfrac{1}{2} < R(m) < \tfrac{1}{2}\ [1]).$$

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ÖZET

n-boyutlu LAPLACE dönüşümleri hakkında bir teorem ispat edildikten sonra, bu teoremin sonuçları, başka usullarla hesapları oldukça karışık bir şekil alan bâzı belirleri integrallere uygulanmakta ve bunların değerleri elde edilmektedir.

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