HOMOGENEOUS NON-STATIC SOLUTIONS OF EINSTEIN-MAXWELL FIELD-EQUATIONS FOR AN ISOTROPIC SPACE-TIME

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We consider solutions of the EINSTEIN-MAXWELL field-equations in vacuo for spatially homogeneous electro-magnetic fields. We find that, except for certain times, the metric is regular everywhere, there is no magnetic-field and there is an electric-field of equal strength in the x, y and z directions which increases with time.

1. **Introduction.** The EINSTEIN-MAXWELL field-equations in vacuo are,

$$G^{\nu}_{\mu} = 8\pi \left(F_{\mu\sigma} F^{\nu\sigma} + \tfrac{1}{4} g^{\nu}_{\mu} F_{\sigma\varepsilon} F^{\sigma\varepsilon}\right), \tag{1.1}$$

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0, \tag{1.2}$$

$$\left(\sqrt{-g}\, F^{\mu\nu}\right)_{,\nu} = 0, \tag{1.3}$$

where G^{ν}_{μ} is the EINSTEIN tensor

$$G^{\nu}_{\mu} \ \underline{\underline{\mathrm{def}}}\ R^{\nu}_{\mu} - \tfrac{1}{2} g^{\nu}_{\mu} R \tag{1.4}$$

where R^{ν}_{μ} is the RICCI curvature tensor

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\tau}_{\sigma\tau} + (\Gamma^{\tau}_{\mu\tau})_{,\nu} - (\Gamma^{\tau}_{\mu\nu})_{,\tau} \tag{1.5}$$

and

$$R \ \underline{\underline{\mathrm{def}}}\ R_{\mu\nu}\, g^{\mu\nu} . \tag{1.6}$$

$g^{\mu\nu}$ is the fundamental tensor and $\Gamma^{\sigma}_{\mu\nu}$ are the CHRISTOFFEL 3-index symbols. $F_{\mu a}$ is the skew-symmetric electromagnetic field tensor.

The RAINICH conditions for the existence of a non-null electro-magnetic field are [1]:

a) The MAXWELL tensor has zero trace,

$$R = 0. \tag{1.7}$$

b) The square of this tensor is proportional to the unit matrix,

$$R^{\sigma}_{\mu} R^{\nu}_{\sigma} = \varrho^{\varrho}\, \delta^{\nu}_{\mu} \tag{1.8}$$

c) The electro-magnetic energy-density is positive definite, (1.9) $$R_{00} > 0.$$

d) A certain form of the RICCI tensor shall have zero curl,

(1.10) $$\frac{g_{\mu\sigma}\, \varepsilon^{\varrho\sigma\nu\lambda}\, R^{\psi}_{\varrho}\, R_{\psi\nu,\lambda}}{4\,\varrho^2 \sqrt{-g}} = a,_{\mu} = 0$$

where $\varepsilon^{\sigma\varrho\nu\lambda}$ is the LEVI-CIVITA permutation symbol and $\varepsilon^{0123} = 1$.

When the EINSTEIN-MAXWELL equations hold, these RAINICH conditions are completely equivalent to EINSTEIN's description of electro-magnetic radiation and gravitation.

Further, for an electro-magnetic field, the eigen values of the EINSTEIN tensor reduce to the form,

(1.11) $$\varrho,\ \varrho,\ -\varrho,\ -\varrho\,.$$

2. **Field-Equations.** We consider the isotropic line-element, [2],

(2.1) $$ds^2 = e^{-2\Phi}\, dt^2 - e^{2\Phi}\,(dx^2 + dy^2 + dz^2)$$

where we take Φ to be a function of t alone.

The components of the EINSTEIN tensor G^{ν}_{μ} for the space-time (2.1) are found to be,

(2 2) $$G^0_0 = 3\, e^{2\Phi}\, \dot{\Phi}^2$$
$$G^1_1 = G^2_2 = G^3_3 = e^{2\Phi}\,(\,5\,\dot{\Phi}^2 + 2\,\dot{\Phi}\,)$$

where the dot denotes differentiation with respect to time.

From (2.2) it is seen that the RAINICH conditions (1.7) to (1.10) are satisfied.

The eigen-values of the EINSTEIN tensor are given by

(2.3) $$G^{\sigma}_{\mu}\, G^{\nu}_{\sigma} = \varrho^2\, g^{\nu}_{\mu}\,.$$

These are seen to satisfy (1.11), with,

(2.4) $$\varrho = 3\, e^{2\Phi}\, \dot{\Phi}^2\,.$$

The condition (1.7) gives,

(2.5) $$\Phi = \frac{1}{3}\log\,(3\,t + c).$$

where c is some constant.

The metric (2.1) now becomes,

(2.6) $$ds^2 = \frac{1}{(3\,t + c)^{2/3}}\, dt^2 - (3\,t + c)^{2/3}\,(dx^2 + dy^2 + dz^2)$$

We see that this metric is regular everywhere except for values of $t = -c/3$.

3. **Solutions.** Working directly with the electromagnetic field tensor we find the solutions of the EINSTEIN-MAXWELL field equations.

From (1.1) and (2.2), we have the following equations :

(3.1) $$g^{22}\,(F^2_{12} + F^2_{13} + F^2_{23}) + g^{00}\,(F^2_{01} + F^2_{02} + F^2_{03}) = \frac{e^{4\Phi}}{4\,\pi}\, 3\,\dot{\Phi}^2$$

(3.2) $$F_{13}\, F_{03} + F_{12}\, F_{02} = 0,$$

(3.3) $$F_{12}\, F_{01} - F_{23}\, F_{03} = 0,$$

(3.4) $$F_{13}\, F_{01} + F_{23}\, F_{04} = 0,$$

(3.5) $$g^{22}\, F_{12}^2 - g^{00}\, F_{03}^2 = \frac{e^{4\Phi}}{4\pi}\,(5\,\dot{\Phi}^2 + 2\,\dot{\Phi}),$$

(3.6) $$g^{22}\, F_{13}^2 - g^{00}\, F_{01}^2 = \frac{e^{4\Phi}}{4\pi}\,(5\,\dot{\Phi}^2 + 2\,\dot{\Phi}),$$

(3.7) $$g_{22}^2\, F_{13}^2 - g^{00}\, F_{02}^2 = \frac{e^{4\Phi}}{4\pi}\,(5\,\dot{\Phi}^2 + 2\,\dot{\Phi}).$$

Equations (3.2) and (3.5) give,

(3.8) $$F_{03}^2\,(g^{22}\, F_{23}^2 - g^{00}\, F_{01}^2) = F_{01}^2\,\frac{e^{4\Phi}}{4\pi}\,(5\,\dot{\Phi}^2 + 2\,\dot{\Phi}).$$

From (3.6) and (3.8), we have,

(3.9) $$F_{03}^2 = F_{01}^2\,.$$

Also, from (3.4), (3.6) and (3.7),

(3.10) $$F_{02}^2 = F_{01}^2\,.$$

From (3.9) and (3.10),

(3.11) $$F_{01}^2 = F_{02}^2 = F_{03}^2\,.$$

Similarly, (3.5), (3.6) and (3.7) give,

(3.12) $$F_{12}^2 = F_{13}^2 = F_{23}^2\,.$$

From (3.11) and (3.12), using (3.1), we have,

(3.13) $$g^{22}\, F_{12}^2 + g^{00}\, F_{03}^2 = \frac{e^{2\Phi}}{4\pi}\,\dot{\Phi}^2.$$

(1.7), (3.5) and (3.13) now give,

(3.14) $$F_{12} = 0$$

and

(3.15) $$F_{03} = \pm\,\frac{\dot{\Phi}\, e^{\Phi}}{\sqrt{\pi}}\,.$$

Hence we have,

(3.16) $$F_{12} = F_{13} = F_{23} = 0$$

and

(3.17) $$F_{01} = F_{02} = F_{03} = \pm\,(3\,t + c)^{4/3}\,.$$

It is seen that, with these values, the equations (1.2) and (1.3) hold.

We find that, if Φ is taken as a function of t alone, there is no magnetic induction but there is an electric field of equal strength in the x, y and z directions. As t increases, the electric field also increases.

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[2] Evidence for Gravitational Theories, Proceedings of the International School of Physics, pp. 237 (1961

ÖZET

EINSTEIN-MAXWELL denklemleri, boşlukta ve uzayca homogen olan elektromagnetik alanlar için göz önüne alınmaktadır. Belirli bazı zaman anlar hariç, metriğin her yerde regüler olduğu, magnelik alanın olmadığı ve x, y ve z doğrultularında eşit kuvveti bulunan ve zaman ile birlikte artan bir elektrik alanının varlığı gösteriliyor.