ELECTROMAGNETIC FIELD IN DECOMPOSABLE SPACE-TIME

V. IYENGAR AND K. KUPPUSWAMY RAO

Considering a four dimensional Riemannian space with signature $-+++$ as the product of two ordinary surfaces with constant curvature, and using the RAINICH algebraic equations we obtain expressions for the electromagnetic field. This is seen to agree with the results obtained by BERTOTTI (1959) on the basis of purely geometric considerations.

1. If we consider space-time as topologically equivalent to the product of two ordinary surfaces Σ_+ (coordinates x^0, x^1) and Σ_- (coordinates x^2, x^3), a tensor field of arbitrary type is said to be decomposable [1] if (*a*) its components with mixed indices are zero and (*b*) if its components relative to Σ_+ depend only on x^0, x^1 and those relative to Σ_- depend only on x^2, x^3. The fundamental tensor $g_{\mu\nu}$ must then be such that

(1.1) $$g_{\mu\nu} = h_{\mu\nu} + f_{\mu\nu}$$

where $h_{\mu\nu}$ pertains to Σ_+ and $f_{\mu\nu}$ to Σ_-.

For source-free electromagnetism with non-null electromagnetic fields, the EINSTEIN-MAXWELL equations are replaced by a set of conditions involving the energy momentum tensor $T_{\mu\nu}$ where

(1.2) $$G_{\mu\nu} \underset{=}{\text{def}} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

where

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu} \Gamma^{\beta}_{\nu\alpha} - \Gamma^{\alpha}_{\beta\alpha} \Gamma^{\beta}_{\mu\nu}$$

and

$$R = R_{\mu\nu} g^{\mu\nu}.$$

The BIANCHI identities give

(1.3) $$G^{\nu}_{\mu;\nu} = 0$$

and then the RAINICH algebraic conditions [2] for a real electromagnetic field are:

The MAXWELL stress tensor has zero trace

(1.4) $$G = 0.$$

The square of the stress tensor is proportional to the unit matrix

(1.5) $$G^{\alpha}_{\mu} \, G^{\nu}_{\alpha} = \varrho^2 \, \delta^{\nu}_{\mu} = \frac{G_{\alpha\beta} \, G^{\alpha\beta}}{4} \, \delta^{\nu}_{\mu}$$

where

$$\varrho^2 \equiv \tfrac{1}{4} \, G^{\mu}_{\alpha} \, G^{\alpha}_{\mu} > 0.$$

The electromagnetic energy density is positive definite

(1.6) $$G_{00} > 0$$

and

(1.7) $$\alpha_{\mu,\nu} - \alpha_{\nu,\mu} = 0$$

where

$$\alpha_{\sigma} \equiv (-g)^{1/2} \, \frac{\varepsilon_{\sigma\nu\lambda\mu} \, G^{\lambda\beta\,;\,\mu} \, G^{\nu}_{\beta}}{\varrho^2}$$

where $\varepsilon_{\sigma\nu\lambda\mu}$ is taken skew-symmetric in all pairs of indices with $\varepsilon_{0123} \equiv 1$.

The matrix T^{ν}_{μ} has the eigenvalues (ϱ, ϱ, $-\varrho$, $-\varrho$) with ϱ a positive scalar defined in a locally Galilean frame by

(1.8) $$\varrho^2 = (\mathbf{H}^2 - \mathbf{E}^2) + (2\,\mathbf{E} \cdot \mathbf{H})^2$$

where **E** and **H** are the electric and magnetic field strengths respectively.

If the skew-symmetric tensor $f_{\mu\nu}$ defines the electromagnetic field, then a tensor $\overset{*}{}_{\mu\nu}$ dual to $f_{\mu\nu}$ is defined by [8]

(1.9) $$f^{*}_{\mu\nu} = \tfrac{1}{2} \, (-g)^{1/2} \, \varepsilon_{\mu\nu\alpha\beta} \, f^{\alpha\beta} .$$

In the MINKOWSKI frame f^* differs from f by the interchange of $\mathbf{E} \rightarrow \mathbf{H}$ and $\mathbf{H} \rightarrow \mathbf{E}$.

If we now define a complex tensor $\omega_{\mu\nu}$ by

(1.10) $$\omega_{\mu\nu} = f_{\mu\nu} + i \, f^{*}_{\mu\nu}$$

then the MAXWELL equations can be written in the differential form as

(1.11) $$\omega^{\nu}_{\mu,\,\nu} = 0$$

(1.12) $$\omega_{[\mu\nu,\,\rho]} = 0$$

or in the integral form as [8]

(1.13) $$\iint_{\mu<\nu} \omega_{\mu\nu} \, d\,(x^{(\mu)}, \, x^{(\nu)}) = 0.$$

Also if k_{μ} and l_{μ} be the null eigen vectors of the RICCI tensor then $f_{\mu\nu}$ is given by

(1.14) $$f_{\mu\nu} = 2\,\varrho^{1/2} \, \{ K_{[\mu} \, l_{\nu]} \, \cos\alpha - \tfrac{1}{2} \, (-g)^{1/2} \, \varepsilon_{\mu\nu\alpha\beta} \, K^{[\mu} \, l^{\nu]} \sin\alpha \}$$

where

$$\operatorname{tg} 2\,\alpha = - \frac{2\,\mathbf{E} \cdot \mathbf{H}}{\mathbf{H}^2 - \mathbf{E}^2} .$$

2. Following BERTOTTI [1] we use a polar frame of reference and choose

$$ds_{+}^{2} = -\left(1 + \frac{x^{2}}{r^{2}}\right) dt^{2} + \left(1 + \frac{x^{2}}{r^{2}}\right)^{-1} dx^{2} \tag{2.1}$$

$$ds_{-}^{2} = \left(1 - \frac{z^{2}}{s^{2}}\right) dy^{2} + \left(1 - \frac{z^{2}}{s^{2}}\right)^{-1} dz^{2} \tag{2.2}$$

where ds_{+} and r refer to Σ_{+}; and ds_{-} and s refer to Σ_{-}. If $r \to s$ and $s \to \infty$ we see that (2.1) and (2.2) reduce to flat space. Denoting by R_{ij} and S_{ij} the RICCI tensors corresponding to Σ_{+} and Σ_{-} respectively and the RICCI tensor for the whole space by G_{ij} we obtain

$$R_{00} = -\frac{1}{r^{2}} \left(1 + \frac{x^{2}}{r^{2}}\right), \tag{2.3}$$

$$R_{11} = \frac{1}{r^{2}} \left(1 + \frac{x^{2}}{r^{2}}\right)^{-1}, \tag{2.4}$$

$$R_{22} = R_{33} = 0 = S_{00} = S_{11}, \tag{2.5}$$

$$S_{22} = -\frac{1}{s^{2}} \left(1 - \frac{z^{2}}{s^{2}}\right), \tag{2.6}$$

$$S_{33} = -\frac{1}{s^{2}} \left(1 - \frac{z^{2}}{s^{2}}\right)^{-1}. \tag{2.7}$$

It follows that

$$G_{ij} = R_{ij} + S_{ij} \tag{2.8}$$

and

$$G = R + S. \tag{2.9}$$

3. From (1.4) we have

$$G_{0}^{0} + G_{1}^{1} + G_{2}^{2} + G_{3}^{3} = 0 \tag{3.1}$$

and from (1.5)

$$(G_{0}^{0})^{2} = (G_{1}^{1})^{2} = (G_{2}^{2})^{2} = (G_{3}^{3})^{2} \tag{3.2}$$

and from (3.1), (3.2) and (2.3) to (2.7) it follows that

$$G_{2}^{2} = G_{3}^{3} \; ; \; G_{0}^{0} = G_{1}^{1} \; ; \; G_{0}^{0} = -G_{2}^{2}. \tag{3.3}$$

We now calculate the null eigenvectors of the RICCI tensor. For $A > 0$, we have

$$R_{\nu}^{\mu} k^{\nu} = -AK^{\mu} \quad , \quad R_{\nu}^{\mu} l^{\nu} = -A l^{\mu}, \tag{3.4}$$

$$k^{\mu} k_{\mu} = 0 \quad , \quad l^{\mu} l_{\mu} = 0 \quad ; \quad k_{\mu} l^{\mu} = -1. \tag{3.5}$$

From (3.4) and (3.5) making use of (3.3) we get for $k^{0} \neq 0$

$$K_{1} = (-g^{00}/g^{11})^{1/2} k_{0}$$

$$K_{2} = K_{3} = 0$$

and

$$l_0 = - {}^{1/2} g^{00} \; k_0 ,$$

$$l_1 = \tfrac{1}{2} \, (-g^{00} / g^{11})^{1/2} \; {}^{1}/g^{00} \; k^0 ,$$

$$l_2 = l_3 = 0 .$$

Thus

(3.6) $$K_\alpha = [\, k_0 , \, (-g^{00} / g^{11})^{1/2} \, k_0 , \, 0 \;\; , \;\; 0 \,] ,$$

(3.7) $$l_\alpha \;\; = [\, -{}^1/_2 \, g^{00} \, k_0 , \, (-g^{00} / g^{11})^{1/2} \; {}^1/_2 \, g^{00} \, k_0 , \; 0 \; , \; 0 \,] .$$

4. From (1.14) the nonvanishing components of f are found to be

(4.1) $$f_{01} = \varrho^{1/2} \cos \alpha ,$$

(4.2) $$f_{23} = \varrho^{1/2} \sin \alpha ,$$

which are constant.

From (1.9) the nonvanishing components of $f^*_{\mu\nu}$ are

(4.3) $$f^*_{01} = \varrho^{1/2} \; \sin \alpha ,$$

(4.4) $$f^*_{23} = - \; \varrho^{1/2} \; \cos \alpha .$$

From (1.10) we get

(4.5) $$\omega_{01} = \varrho^{1/2} \; e^{i\alpha} ,$$

(4.6) $$\omega_{23} = - \, i \; \varrho^{1/2} \; e^{i\alpha} .$$

Since from (1.5)

$$\varrho^{1/2} = (\, G_0^0 \,)^{1/2}$$

(4.5) and (4.6) now become

(4.7) $$\omega_{01} = (\, G_0^0)^{1/2} \; e^{i\alpha}$$

(4.8) $$\omega_{23} = - \, i \, (\, G_0^0)^{1/2} \; e^{i\alpha}$$

It can easily be verified that these values of $\omega_{\mu\nu}$ satisfy the MAXWELL equations (1.11) and (1.12) as well as (1.13).

The electromagnetic field given by (4.1) and (4.2) is the same as obtained by BERTOTTI [1].

REFERENCES

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OSMANIA UNIVERSITY
HYDERABAD (INDIA).

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ÖZET

İmzası — + + + şeklinde olan dört boyutlu RIEMANN uzayı, sâbit eğriliği hâiz iki yüzeyin çarpımı gibi düşünülerek, RAINICH'in cebirsel bağıntılarını uygulamak suretiyle elektromanyetik alan için ifadeler elde ediliyor. Bunların, 1959 yılında, BERTOTTI tarafından, sâdece geometrik mülâhazalara dayanarak elde edilen ifadelere uydukları görülüyor.