ON RELATIVE CURVATURE TENSORS IN THE SUBSPACE OF A RIEMANNIAN SPACE

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A connection parameter, whether symmetric or asymmetric, may be used to define certain methods of covariant differentiation, parallelisms and the curvature tensors of the space for which the connection has been defined. The relative connection coefficient (given in [3]) defines two processes of covariant differentiation. Pseudo-geodesics are auto-parallel curves relative to each of the two methods of differentiation. PRVANOVITCH [3] has used this connection parameter in the derivation of what have been called the relative FRENET formulae. In the present paper the author has obtained three curvature tensors called the relative curvature tensors of the first, second and third kinds. A number of identities have been deduced. The notion of relatively flat subspaces has been introduced for each process of differentiation.

1. **Fundamental formulae.** Let a subspace, V_n, given by the equations

$$y^\alpha = y^\alpha(x^i) \quad ; \quad \alpha = 1, \dots, m \quad ; \quad i = 1, \dots, n \; ;$$

be immersed in an m-dimensional riemannian space V_m. Consider a set of unit vectors $\lambda^\alpha_{(\mu)}$ $(\mu = n+1, \dots, m)$ defining $(m-n)$ congruences of curves such that through each point of V_m there passes exactly one curve of each congruence. At the points of the subspace, we may write

$$\lambda^\alpha_{(\mu)} = t^i_{(\mu)} B^\alpha_i + \sum_{\nu=n+1}^{m} C_{(\mu\nu)} N^\alpha_{(\nu)} \quad , \quad (\mu = n+1, \dots, m), \tag{1.1}$$

where the components $t^i_{(\mu)}$ and the scalars $C_{(\mu\nu)}$ are functions of x^i, $N^\alpha_{(\nu)}$ are unit normal vectors of the subspace and $B^\alpha_i = \dfrac{\partial y^\alpha}{\partial x^i}$. It is assumed that the vectors $\lambda^\alpha_{(\mu)}$, together with any n linearly independent vectors of V_n, form a set of m linearly independent vectors of V_m. This is possible if and only if the determinant $|C_{(\mu\nu)}| \neq 0$.

Consider a vector field $\xi^i = \xi^i(x)$ of the subspace. With respect to this vector field, an asymmetric connection M^i_{jk}, called *relative connection parameter*, has been defined in [3]. This is given by

$$M^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk} \, , \tag{1.2}$$

where

$$\Omega^i_{jk} = \sum_{\mu} \sum_{\nu} \bar{C}_{(\mu\nu)} \, k_{(\nu)} \, (\delta^i_k \, \delta^r_h - \delta^i_h \, \delta^r_k) \, g_{jr} \, t^h_{(\mu)} \tag{1.3}$$

is a tensor, Γ^i_{jk} is the CHRISTOFFEL symbol of the second kind, $\bar{C}_{(\mu\nu)}$ is the cofactor of $C_{(\mu\nu)}$ in $| C_{(\mu\nu)} | \, / \, | C_{(\mu\nu)} |$ and $k_{(\nu)} = \Omega_{(\nu)\, ij} \, \xi^i \, \xi^j$, $\Omega_{(\nu)\, ij}$ being the second fundamental tensors of the subspace. A pseudogeodesic ([1] and [2]) of the subspace is given by

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} - \sum_{\mu} \sum_{\nu} \bar{C}_{(\mu\nu)} \, \Omega_{(\mu)\, lj} \frac{dx^l}{ds} \frac{dx^j}{ds} \times \left(t^i_{(\nu)} - t_{(\nu)\, j} \frac{dx^j}{ds} \frac{dx^i}{ds} \right) = 0 . \tag{1.4}$$

2. Relative covariant differentiation. The connection parameter M^i_{jk} may be used in the following two definitions of relative covariant differentiation of a tensor of type $T^i_j \, (x \, , \xi)$.

$$T^i_{j|k} \overset{\text{def}}{=} \frac{\partial \, T^i_j}{\partial x^k} + \frac{\partial \, T^i_j}{\partial \xi^d} \frac{\partial \, \xi^d}{\partial x^k} + T^r_j \, M^i_{rk} - T^i_r \, M^r_{jk} \tag{2.1}$$

and

$$T^i_{j;k} \overset{\text{def}}{=} \frac{\partial \, T^i_j}{\partial x^k} + \frac{\partial \, T^i_j}{\partial \xi^d} \frac{\partial \, \xi^d}{\partial x^k} + T^r_j \, M^i_{kr} - T^i_r \, M^r_{kj} \, . \tag{2.2}$$

It may be verified that the curve $x^i = x^i \, (s)$ is a pseudogeodesic of the subspace if we have

$$\left(\frac{dx^i}{ds} \right)_{|j} \frac{dx^j}{ds} = 0, \quad \text{or} \quad \left(\frac{dx^i}{ds} \right)_{;j} \frac{dx^j}{ds} = 0 \, .$$

Therefore, pseudogeodesics are auto-parallel with respect to each of the two definitions of relative covariant differentiation.

It may be verified that

$$\text{(a)} \quad \Omega_{ijk} + \Omega_{jik} = 0 \, , \qquad \text{(b)} \quad g_{ij\,|\,k} = 0 \tag{2.3}$$

and

$$\text{(c)} \qquad g_{ij\,;\,k} = \sum_{\mu} D_{(\mu)} \, (t_{(\mu)\, i} \, g_{jk} + t_{(\mu)\, j} \, g_{ik} - 2 \, g_{ij} \, t_{(\mu)\, k}) \, ,$$

where

$$\Omega_{ijk} = g_{ir} \, \Omega^r_{jk} \quad \text{and} \quad D_{(\mu)} = \sum_{\nu} \bar{C}_{(\mu\nu)} \, k_{(\nu)} \, .$$

3. Relative curvature tensor of the first kind. Consider a vector field $v^i \, (x)$ of the subspace. Subjecting this to the relative covariant differentiation of the type (2.1), we get

$$v^i_{|jk} = \frac{\partial^2 v^i}{\partial x^k \partial x^j} + \frac{\partial v^l}{\partial x^k} M^i_{lj} + \frac{\partial v^l}{\partial x^j} M^i_{lk} + v^l \left(\frac{\partial M^i_{lj}}{\partial x^k} + \frac{\partial M^i_{lj}}{\partial \xi^d} \frac{\partial \xi^d}{\partial x^k} + M^r_{lj} M^i_{rk} \right) - v^i_{|r} M^r_{jk}, \tag{3.1}$$

whence

$$v^i_{|jk} - v^i_{|kj} = v^l \underset{1}{\widehat{R}}{}^i_{.lkj} - v^i_{|r} (\Omega^r_{jk} - \Omega^r_{kj}), \tag{3.2}$$

where

$$\underset{1}{\widehat{R}}{}^i_{.jhk} = \frac{\partial M^i_{jk}}{\partial x^h} + \frac{\partial M^i_{jk}}{\partial \xi^d} \frac{\partial \xi^d}{\partial x^h} - \frac{\partial M^i_{jh}}{\partial x^k} - \frac{\partial M^i_{jh}}{\partial \xi^d} \frac{\partial \xi^d}{\partial x^k} + M^i_{rh} M^r_{jk} - M^i_{rk} M^r_{jh}. \tag{3.3}$$

Remark. It may be noted that the term $v^i_{|r} (\Omega^r_{jk} - \Omega^r_{kj})$ occuring in the right hand side of (3.2) includes v^l. Its coefficient has not been absorbed in $\underset{1}{\widehat{R}}{}^i_{.lkj}$ as the altered value of the relative curvature tensor will not be consistent with a theorem (Theorem (6.1)) on the existence of relatively parallel vector fields.

Using the fact that the CHRISTOFFEL symbol Γ^i_{jk} is independent of ξ^i and substituting from (1.2) the equation (3.3) simplfies to

$$\underset{1}{\widehat{R}}{}^i_{.jhk} = K^i_{.jhk} + \Omega^i_{jk\,;\,h} - \Omega^i_{jh\,;\,k} + \Omega^i_{rh} \Omega^r_{jk} - \Omega^i_{rk} \Omega^r_{jh}, \tag{3.4}$$

where $K^i_{.jhk}$ is the RIEMANN (curvature) tensor of the subspace.

The tensor $\underset{1}{\widehat{R}}{}^i_{.jhk}$ is called the relative curvature tensor of the first kind and we have

$$\underset{1}{\widehat{R}}_{ijhk} \overset{\text{def}}{=} g_{ir} \underset{1}{\widehat{R}}{}^r_{.jhk} = K_{ijhk} + \Omega_{ijk\,;\,h} - \Omega_{ijh\,;\,k} + \Omega_{irh} \Omega^r_{jk} - \Omega_{irk} \Omega^r_{jh}. \tag{3.5}$$

The following results are obvious :

$$\underset{1}{\widehat{R}}_{ijhk} = - \underset{1}{\widehat{R}}_{ijkh} \quad \text{and} \quad \underset{1}{\widehat{R}}{}^i_{.jhk} = - \underset{1}{\widehat{R}}{}^i_{.jkh}. \tag{3.6}$$

Introducing certain interchanges of indices and using the relation (2.3) (a), we get

$$\Omega_{irh} \Omega^r_{jk} - \Omega_{irk} \Omega^r_{jh} = - (\Omega_{jrh} \Omega^r_{ik} - \Omega_{jrk} \Omega^r_{ih}).$$

This relation, the use of (2.3) (a) once again and the fact that

$$K_{ijhk} = - K_{jihk}$$

proves

$$\underset{1}{\widehat{R}}_{ijhk} = - \underset{1}{\widehat{R}}_{jihk}. \tag{3.7}$$

Finally, from (3.5) we obtain

$$\underset{1}{\widehat{R}}_{ijhk} + \underset{1}{\widehat{R}}_{ihkj} + \underset{1}{\widehat{R}}_{ikjh} = \Omega_{i\overline{jk}\,;\,h} + \Omega_{i\overline{hj}\,;\,k} + \Omega_{i\overline{kh}\,;\,j} \tag{3.8}$$

$$+ \left(\Omega_{irk}\, \Omega^{r}_{\overline{hj}} + \Omega_{irh}\, \Omega^{r}_{\overline{jk}} + \Omega_{iri}\, \Omega^{r}_{\overline{kh}}\right),$$

where

$$\Omega_{i\overline{jh}} = \Omega_{ijh} - \Omega_{ihj} \quad \text{and} \quad \Omega^{i}_{\overline{jh}} = \Omega^{i}_{jh} - \Omega^{i}_{hj}$$

and we have used the following well known identity about the RIEMANN tensor [1, 112]

$$K_{ijhk} + K_{ihkj} + K_{ikjh} = 0. \tag{3.9}$$

Using the equation (1.3) it can be proved that the expression in brackets on the right hand side of (3.8) vanishes and we have

$$\underset{1}{\widehat{R}}_{ijhk} + \underset{1}{\widehat{R}}_{ihkj} + \underset{1}{\widehat{R}}_{ikjh} = \Omega_{i\overline{jk}\,;\,h} + \Omega_{i\overline{hj}\,;\,k} + \Omega_{i\overline{kh}\,;\,j}\,. \tag{3.10}$$

4. Relative curvature tensor of the second kind. Consider the relative covariant differentiation of the type given by (2.2). Proceeding as in section 3 we deduce

$$v^{i}_{\,;\,jk} - v^{i}_{\,;\,kj} = v^{l}\, \underset{2}{\widehat{R}}{}^{i}_{\,.\,lkj} - v^{i}_{\,.\,|\,r}\,(\Omega^{r}_{kj} - \Omega^{r}_{jk}), \tag{4.1}$$

where

$$\underset{2}{\widehat{R}}{}^{i}_{\,.\,jhk} = K^{i}_{\,.\,jhk} + \Omega^{i}_{kj\,:\,h} - \Omega^{i}_{hj\,;\,k} + \Omega^{i}_{hr}\, \Omega^{r}_{kj} - \Omega^{i}_{kr}\, \Omega^{r}_{hj}\,. \tag{4.2}$$

The tensor $\underset{2}{\widehat{R}}{}_{\,.\,jhk}$ is called the relative curvature tensor of the second kind. Also we have

$$\underset{2}{\widehat{R}}_{ijhk} \overset{\text{def}}{=} g_{ir}\, \underset{2}{\widehat{R}}{}^{r}_{\,.\,jhk} = K_{ijhk} + \Omega_{ikj\,;\,h} - \Omega_{ihj\,;\,k} \tag{4.3}$$

$$+ \left(\Omega_{ihr}\, \Omega^{r}_{kj} - \Omega_{ikr}\, \Omega^{r}_{hj}\right).$$

It is obvious that

$$\underset{2}{\widehat{R}}_{ijhk} = -\underset{2}{\widehat{R}}_{ijkh} \quad \text{and} \quad \underset{2}{\widehat{R}}{}^{i}_{\,.\,jhk} = -\underset{2}{\widehat{R}}{}^{i}_{\,.\,jkh}\,. \tag{4.4}$$

However, in contradiction to $\underset{1}{\widehat{R}}_{ijhk}$ or K_{ijhk}, the tensor $\underset{2}{\widehat{R}}_{ijhk}$ is *not* skew symmetric in the first two indices.

The equations (4.3), (3.9) and (1.3) yield

$$\underset{2}{\widehat{R}}_{ijhk} + \underset{2}{\widehat{R}}_{ihkj} + \underset{2}{\widehat{R}}_{ikjh} = \Omega_{i\overline{kj}\,;\,h} + \Omega_{i\overline{jh}\,;\,k} - + \Omega_{i\overline{hk}\,;\,j}\,. \tag{4.5}$$

5. Relative curvature tensor of the third kind. Using the covariant derivatives of the types given by equations (2.1) and (2.2) it may be verified that

$$v^i_{|j\,;\,k} = \frac{\partial^2 v^i}{\partial x^k\,\partial x^j} + v^l\left(\frac{\partial M^i_{lj}}{\partial x^k} + \frac{\partial M^i_{lj}}{\partial \xi^d}\,\frac{\partial \xi^d}{\partial x^k} + M^r_{lj}\,M^i_{kr}\right) + \frac{\partial v^r}{\partial x^k}\,M^i_{rj} + \frac{\partial v^r}{\partial x^j}\,M^i_{kr} - v^i_{\,|\,r}\,M^r_{kj}\,. \tag{5.1}$$

Since

$$v^i_{\,|\,r} - v^i_{\,;\,r} = v^l\,(M^i_{lr} - M^i_{rl})\,, \tag{5.2}$$

it follows {from (5.1) and a similar equation},

$$v^i_{\,|\,j\,.\,|\,k} - v^i_{\,.\,|\,k\,|\,j} = v^l\,\underset{3}{\widehat{R}}{}^i_{\,.\,lkj}\,, \tag{5.3}$$

where

$$\underset{3}{\widehat{R}}{}^i_{\,.\,jhk} = \frac{\partial M^i_{jk}}{\partial x^h} + \frac{\partial M^i_{jk}}{\partial \xi^d}\,\frac{\partial \xi^d}{\partial x^h} - \frac{\partial M^i_{hj}}{\partial x^k} - \frac{\partial M^i_{hj}}{\partial \xi^d}\,\frac{\partial \xi^d}{\partial x^k} + M^i_{hr}\,M^r_{jk} - M^i_{rk}\,M^r_{hj} - M^i_{jr}\,M^r_{hk} + M^i_{rj}\,M^r_{hk}\,. \tag{5.4}$$

The above equation, when simplified with the help of (1.2), reduces to

$$\underset{3}{\widehat{R}}{}^i_{\,.\,jhk} = K^i_{\,.\,jhk} + \Omega^i_{jk\,,\,h} - \Omega^i_{hj\,;\,k} + \Omega^i_{hr}\,\Omega^r_{jk} - \Omega^i_{rk}\,\Omega^r_{hj} - \Omega^i_{jr}\,\Omega^r_{hk} + \Omega^i_{rj}\,\Omega^r_{hh}\,. \tag{5.5}$$

The tensor $\underset{3}{\widehat{R}}{}^i_{\,.\,jhk}$ is called the relative curvature tensor of the third kind. This is *not* skew symmetric in the indices h and k. We have

$$\underset{3}{\widehat{R}}_{ijhk} \overset{\text{def}}{=} g_{ir}\,\underset{3}{\widehat{R}}{}^r_{\,.\,jhk} = K_{ijhk} + \Omega_{ijk\,;\,h} - \Omega_{ihj\,;\,k} + \Omega_{ihr}\,\Omega^r_{jk} - \Omega_{irk}\,\Omega^r_{hj} - \Omega_{ijr}\,\Omega^r_{hk} + \Omega_{irj}\,\Omega^r_{hk}\,. \tag{5.6}$$

It may be verified from the equations (1.3), (3.9) and (5.6) that

$$\underset{3}{\widehat{R}}_{ijhk} + \underset{3}{\widehat{R}}_{ihkj} + \underset{3}{\widehat{R}}_{ikjh} = 0\,. \tag{5.7}$$

A simple calculation based on the equations (3.5), (4.3) and (5.6) yields

$$\underset{1}{\widehat{R}}_{ijhk} + \underset{2}{\widehat{R}}_{ihkj} + \underset{3}{\widehat{R}}_{ikjh} = 0\,. \tag{5.8}$$

The following commutation formulae may be of some interest :

$$v_{i\,|\,jk} - v_{i\,|\,kj} = -\,v_l\,\underset{1}{\widehat{R}}{}^l_{\,.\,ikj} - v_{i\,|\,r}\,(\Omega^r_{jk} - \Omega^r_{kj})\,, \tag{5.9}$$

$$v_{i\,.\,|\,jk} - v_{i\,.\,|\,kj} = -\,v_l\,\underset{2}{\widehat{R}}{}^l_{\,.\,ihj} - v_{i\,.\,|\,r}\,(\Omega^r_{kj} - \Omega^r_{jk}) \tag{5.10}$$

and

$$v_{i\,|\,j\,.\,k} - v_{i\,.\,|\,k\,|\,j} = - v_l \underset{3}{\widehat{R}}{}^{l}_{.\,ikj} \,. \tag{5.11}$$

6. Relatively flat subspaces. Definition (6.1). A subspace V_n is said to be a relatively flat subspace of the first or second or third kind according as

$$\underset{1}{\widehat{R}}_{ijhk} = 0 \quad \text{or} \quad \underset{2}{\widehat{R}}_{ijhk} = 0 \quad \text{or} \quad \underset{3}{\widehat{R}}_{ijhk} = 0 \,, \tag{6.1}$$

at all the points of V_n. It is said to be a relatively flat subspace if all the three conditions given in (6.1) are satisfied.

Definition (6.2). A vector field v^i is called a relatively parallel vector field of the first kind if

$$v^i{}_{|\,j} = 0 \,, \qquad \text{identically.} \tag{6.2 a}$$

On the other hand it is said to be a relatively parallel vector field of the second kind if

$$v^i{}_{.\,|\,j} = 0 \,, \qquad \text{at all the points of } V_n . \tag{6.2 b}$$

We shall prove the following :

Theorem (6.1).

A necessary and sufficient condition that a V_n admits a set of n linearly independent relatively parallel vector fields of the first kind is that it is a relatively flat subspace of the first kind.

Proof. The equation (6.2) a may be written as

$$\left(\frac{\partial v^i}{\partial x^j}\right) = - M^i_{hj}\, v^h \,. \tag{6.3}$$

The conditions,

$$\left(\frac{\partial^2 v^i}{\partial x^k\, \partial x^j}\right) = \left(\frac{\partial^2 v^i}{\partial x^j\, \partial x^k}\right),$$

of integrability of (6.3) are equivalent to

$$v^h \underset{1}{\widehat{R}}{}^{i}_{.\,hjk} = 0 \,. \tag{6.4}$$

Theorem (6.1) is immediate from this equation.

Theorem (6.2).

A necessary and sufficient condition that a V_n admits a set of n linearly independent relatively parallel vector fields of the second kind is that it is a relatively flat subspace of the second kind.

The proof is similar to that of Theorem (6.1).

Theorem (6.3).

If a V_n admits n linearly independent relatively parallel vector fields of the first kind and also n linearly independent relatively parallel vector fields of the second kind then it is a relatively flat subspace.

The proof of this theorem is an immediate consequence of Theorems (6.1), (6.2) and the equation (5.8).

The relative curvature tensors studied in this paper may be used in the definition of scalars corresponding to Riemannian curvature of the subspace and also in the derivation of generalised GAUSS characteristic equations of the subspace. These properties will be studied in a separate paper.

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ÖZET

Herhangi bir uzayda simetrik veya asimetrik olabilen bir bağımlılık parametresi verildikten sonra, bu parametre sâyesinde bir kovaryant türev alma işlemi, bir paralellik kavramı ve bir eğrilik tensörü tanımlanabilir. PRVANOVITCH [3], rölatif bağımlılık parametresi dediği böyle bir parametre tanımlamış ve bunu rölatif FRENET formülleri olarak adlandırılan bağıntıları elde etmek için kullanmıştır. Bu araştırmada bu parametre ile farklı iki kovaryant türev alma işlemi tanımlanmıştır. Bu iki türev işlemine göre kendi kendine paralel olma özelliğini hâiz eğriler psödogeodezik eğrilerdir. Ayrıca, yine bu türev alma işlemlerini uyguluyarak, birinci, ikinci ve üçüncü cins rölatif eğrilik tensörleri denen üç çeşit tensör tanımlanmış ve bu tensörlerin sağladıkları bâzı özdeşlikler elde edilmiştir. Her türev alma işlemi için bir rölatif düz alt uzay kavramı da tanımlanmıştır.