NUCLEAR AND STRONGLY NUCLEAR SEQUENCE SPACES

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In this article the relationships between nuclearity and strong nuclearity of smooth sequence spaces have been examined. It has also been shown that a strongly nuclear sequence space is co-nuclear. Further, a theorem on the structure of a strongly nuclear sequence space has been proved.

1 — Terminology

Let P be a set of sequences of non-negative real numbers with the following properties:

(*i*) for every integer m there exists a sequence $\mathrm{a} = (a_n)$ in P with $a_m > 0$.

(*ii*) for every finite collection of sequences $\mathrm{a}^1, \ldots, \mathrm{a}^k$ from P, there exists a sequence a in P with $a_n \geq \max \{ a_n^1, \ldots, a_n^k \}$ for every positive integer n. P is then called a system of steps and the space of all sequences of complex numbers (ξ_n) which satisfy

$$P_a (\xi_n) = \sum_{n=1}^{\infty} | \xi_n | a_n < \infty \tag{1}$$

for every $\mathrm{a} = (a_n)$ in P is called the sequence space generated by P. We denote it by $\lambda(P)$. $\lambda(P)$ becomes a complete locally convex space under the topology generated by the semi-norms defined by (1). A sequence space $\lambda(P)$ is nuclear if and only if for every a in P there exists $\mathbf{b}$ in P with

$$\sum_{n=1}^{\infty} a_n / b_n < \infty \tag{2}$$

where we use the convention that $a_n / b_n = 0$ if $b_n = 0$. (2) is called the GROTHENDIECK-PIETSCH condition. Similarly, $\lambda(P)$ is strongly nuclear if and only if for every a in P there exists $\mathbf{b}$ in P such that the sequence (a_n / b_n) is rapidly decreasing; *i.e* $(n^k a_n / b_n)$ is bounded for each $k = 1, 2, 3\ldots$ (See [2]). Strongly nuclear spaces were first defined and examined by BRUDOVSKII in [1].

Let P be a system of steps with the additional conditions :

(*iii*) each a in P is increasing and strictly positive.

(*iv*) for each a in P there exists b in P with $(a_n)^2 \leq b_n$ for every n.

Then $\lambda(P)$ is called a smooth sequence space of infinite type, and we use the abbreviation G_∞ - space. Similarly, if Q is a system of steps with

(*iii*)′ each $\mathbf{q}$ in Q is decreasing and strictly positive

(*iv*)′ for each $\mathbf{q}$ in Q there exists $\vec{\varrho}$ in Q with $\sqrt{q_n} \leq \varrho_n$ for every n, then $\lambda(Q)$ is called a smooth sequence space of finite type and we write a G_1 - space.

For an increasing sequence (β_n) of non-negative real numbers we denote by $\lambda_\infty(\beta_n)$ the sequence space generated by the system of steps $\{(\varrho^{\beta_n}) : 0 < \varrho\}$ and by $\lambda_1(\beta_n)$ the sequence space generated by $\{(q^{\beta_n}) : 0 < q < 1\}$. $\lambda_\infty(\beta_n)$ is called a power series space of infinite type and $\lambda_1(\beta_n)$ a power series space of finite type. Every power series space of infinite respectively of finite type is a G_∞ - respectively a G_1 - space. The power series space $\lambda_\infty(\log n)$ is the space of all rapidly decreasing sequences and it is denoted by s. This space can be generated also by the system $\{(n^k) : k = 1, 2, \dots\}$.

2 — Smooth sequence spaces

It is known that a smooth sequence space $\lambda(P)$ of infinite type is nuclear if and only if P contains a sequence a $= (a_n)$ with

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty.$$

(See [⁶]) We show that a similar condition is necessary and sufficient for strong nuclearity.

(1) *A G_∞ - space $\lambda(P)$ is strongly nuclear if and only if there exists a sequence* a *in P with $(1/a_n)$ rapidly decreasing.*

Proof. If $\lambda(P)$ is strongly nuclear, then for every b in P there exists a in P with (b_n/a_n) rapidly decreasing. Since $b_0/a_n \leq b_n/a_n$ we have that $(1/a_n)$ is also rapidly decreasing.

On the other hand, if there exists a in P with $(1/a_n)$ rapidly decreasing, then for any b in P we choose d from P with $d_n \geq \max(b_n, a_n)$ and another sequence c in P with $c_n \geq d_n^2$. Then $(1/d_n)$ is rapidly decreasing and hence from the inequality

$$b_n / c_n \leq b_n / d_n^2 \leq 1 / d_n$$

we have that (b_n/c_n) is rapidly decreasing, which shows that $\lambda(P)$ is strongly nuclear.

An immediate consequence of (1) is

(2) *A power series space $\lambda_\infty(\beta_n)$ is strongly nuclear if and only if $(q_0^{\beta_n})$ is rapidly decreasing for some q_0 with $0 < q_0 < 1$.*

For smooth sequence spaces of finite type we get a result which is stronger than (1). First we need a simple Lemma.

Lemma. *A decreasing sequence (q_n) of positive real numbers is rapidly decreasing if and only if*

$$\sum_{n=1}^{\infty} (q_n)^\lambda < \infty$$

for every real number $\lambda > 0$.

Proof. If (q_n) is rapidly decreasing, for a given $\lambda > 0$ we choose an integer k with $\lambda k > 2$. Since $n^k q_n \leq M$ for some $M > 0$ we have $q_n^\lambda \leq \dfrac{M^\lambda}{n^{k\lambda}}$ and so $\Sigma\, q_n^\lambda < \infty$.

If on the other hand $\Sigma\, q_n^\lambda < \infty$ for every $\lambda > 0$, then for a given integer k we choose $\lambda = \dfrac{1}{k}$. Since (q_n) is a decreasing sequence, the inequality

$$nq_n^\lambda \leq \sum_{m=1}^{n} q_m^\lambda \leq \sum_{m=1}^{\infty} q_m^\lambda = M < \infty$$

holds and so we have $n^k q_n \leq M^k$ for every n.

(3) *For a G_1 - space $\lambda(Q)$ the following conditions are equivalent :*

(*a*) $\lambda(Q)$ *is strongly nuclear*

(*b*) $\lambda(Q)$ *is nuclear*

(*c*) $Q \subset l^1$

(*d*) $Q \subset s$

Proof. The pattern of the proof is $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (a)$. Of these, the first implication is trivial.

$(b) \Longrightarrow (c)$: By GROTHENDIECK-PIETSCH condition for every sequence $\mathbf{q}$ in Q, there exist sequences $\mathbf{q}'$ in Q and (ξ_n) in l^1 with $q_n \leq \xi_n q'_n$. Since $\mathbf{q}'$ is decreasing, we have $q_n \leq \xi_n q'_n$ and so $\mathbf{q}$ is in l^1.

$(c) \Longrightarrow (d)$: Let $\mathbf{q}$ be a sequence from Q and $\lambda > 0$. We choose $k \geq 1$ with $2^{-k} \leq \lambda$ and $\mathbf{q}'$ from Q with $q'_n \geq (q_n)\, 2^{-k}$. Then from the inequality

$$(q_n)^\lambda = (q_n^{2^{-k}})^{\lambda 2^k} \leq (q'_n)^{\lambda 2^k}$$

it follows that

$$\sum_{n=1}^{\infty} q_n^\lambda < \infty,$$

since $\lambda\, 2^k \geq 1$ and

$$\sum_{n=1}^{\infty} q'_n < \infty$$

by assumption. Hence by our Lemma $\mathbf{q}$ belongs to s.

$(d) \Rightarrow (a)$: For any $\mathbf{q}$ in Q we choose $\mathbf{q}'$ from Q with $\sqrt{q_n} \leq q'_n$. Then from $q_n \leq q'_n\, q'_n$ we have that $\lambda(Q)$ is strongly nuclear, since by assumption the sequence (q'_n) is rapidly decreasing.

Next we apply our results to power series spaces.

(4) *For an increasing sequence (β_n) of non-negative real numbers the following conditions are equivalent :*

(*a*) $\lambda_\infty(\beta_n)$ *is strongly nuclear*

(*b*) $\lambda_1(\beta_n)$ *is nuclear*

(*c*) $\lambda_1(\beta_n)$ *is strongly nuclear*

Proof. $(b) \Longrightarrow (c)$ is an immediate consequence of (3) and $(c) \Longrightarrow (a)$ follows from (3), (*d*) and (2). It remains to prove $(a) \Longrightarrow (b)$. By (2) we can find a number q_0, $0 < q_0 < 1$ with $(q_0^{\beta_n})$ rapidly decreasing. For a number q, $0 < q < 1$, we choose $\lambda > 0$ with $q_0^\lambda \geq q$. Then $(q_0^\lambda)^{\beta_n} \geq q^{\beta_n}$ and from the Lemma we have then

$$\sum_{n=1}^{\infty} q^{\beta_n} < \infty.$$

It follows from (3), (*e*) that $\lambda_1(\beta_n)$ is nuclear.

Nuclearity and strongly nuclearity of G_∞ - spaces are not equivalent; for s is a nuclear power series space of infinite type which is not strongly nuclear.

3 — Structure of strongly nuclear spaces.

KÖTHE showed in [1] that a nuclear sequence space $\lambda(P)$ is co-nuclcar if and only if the positive sequences in $\lambda(P)$ satisfy the GROTHENDIECK-PIETSCH condition. He also asked the question whether every nuclear sequence space is co-nuclear. We show that this is true for strongly nuclear spaces.

(1) *If a sequence space λ (P) is strongly nuclear, then it is also co-nuclear.*

Proof. Let (ξ_n) be a sequence in $\lambda(P)$ with $\xi_n \geqq 0$. For any a in P we choose **b** from P with (a_n / b_n) rapidly decreasing. Hence we can find $M > 0$ with $n^2 a_n \leqq M b_n$. So from the inequality

$$\sum_{n=1}^{\infty} n^2 \; {}_n a_n \geqq M \sum_{n=1}^{\infty} \xi_n b_n < \infty$$

it follows that $(n^2 \xi_n)$ is in $\lambda(P)$. If $\eta_n = n^2 \xi_n$, then $\xi_n \leqq \frac{1}{n^2} \eta_n$ aud so $\lambda(P)$ is co-nuclear.

This result is not true in general; for ω_d is strongly nuclear as a topological product of strongly nuclear spaces, but its strong dual φ_d is not nuclear for $d > x_0$. The converse of (1) is also false. s' is nuclear (in fact strongly nuclear) but s is not strongly nuclear. This example also shows that the duality between nuclear F- and DF- spaces breaks down in the more restrictive case of strongly nuclear spaces.

We now start to prove a theorem on the structures of strongly nuclear sequence spaces which is similar to the theorem on the structures of nuclear sequence spaces in [1].

Let $\lambda(P)$ be a sequence space. If we add to P all sequences λ a, a in P and $\lambda > 0$, and all positive sequences **b** with $b_n \leqq a_n$ for some a in P, the locally convex space $\lambda(P)$ remains the same. Let us assume then that a system of steps P satisfies the following additional properties :

(*iii*) if a $\in P$, then λ a $\in P$ for every $\lambda > 0$.

(*iv*) if a $\in P$, then **b** $\in P$ for every sequence **b** with $0 \leqq b_n \leqq a_n$ for every n.

The dual λ' of λ (P) is then the normal hull of P.

(2) *If λ (P) is strongly nuclear, then for each* a *in P, the vectors $(n^p a_n)$, $p = 1, 2, \ldots$, are also in P.*

Proof. By assumption there exists b in P with (a_n / b_n) rapidly decreasing. Hence for every $p = 1, 2, \ldots$ we can find $\lambda_p > 0$ with $n^p a_n \leqq \lambda_p b_n$ for every n.

If a is a strictly positive vector in P, then we denote by λ' (a) the normal hull of the set $\{(n^p a_n) : p = 1, 2, \ldots\}$. By (2) λ' (a) is a subspace of λ and it is equal to s' up to the diagonal transformation which sends a to the vector (1, 1, 1, ...). If a from P is not strictly positive, then we define $M(a) = \{n : a_n > 0\}$ and $N(a) = \{n : a_n = 0\}$. If $M(a)$ is finite, we set $\lambda'(a) = \varphi$; if infinite we apply our previous consideration to sequences which have indices only in $M(a)$ and thus get a space λ'_M (a) which is identical with s' up to a diagonal transformation. On $N(a)$ we denote by φ (a) the space of all sequences with only a finite number of non-zero coordinates and with indices in $N(a)$ only. In this case we denote by λ' (a) the sum λ'_M (a) $\oplus$ φ (a). Since $\lambda' = (\lambda(P))'$ is the normal hull of P, we have $\lambda' = \bigcup_P \lambda'$ (a). By taking the α - dual of λ we get.

(3) *Every strongly nuclear space* $\lambda(P)$ *is the intersection* $\bigcap_P \lambda$ (a) *where* λ (a) *is equal to* ω *or to* λ_M (a) $\oplus$ ω (a) *where* λ_M (a) *is isomorphic to s by a diagonal transformation and* ω (a) *is the space of all sequences with indices in* $N(a)$.

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ÖZET

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