ON THE SEQUENCE OF STRAIGHT LINES FORMED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH A RULED SURFACE

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In this paper we have studied the sequence of straight lines which are formed hy the instantaneous screwing axes of the trihedrons associated with a given ruled surface ; furthermore we have examined the infinitesimal elements of a ruled surface and its approximation of order n in the neighbourhood of one of its generators, since they are closely connected with our subject.

1. Let x be a variable point on a space curve (x) and let $x^{(1)}$ denote the center of the osculating sphere at that point. As x traces out the curve (x), in general the point $x^{(1)}$ traces out another space curve, say $(x^{(1)})$. The center of the osculating sphere at a point $x^{(1)}$ on $(x^{(1)})$ will be denoted by $x^{(2)}$. Continuing in this manner, the center of the osculating sphere at a point $x^{(n-1)}$ on $(x^{(n-1)})$ will be $x^{(n)}$. Therefore to every point x on (x) there corresponds a sequence of points, namely $x^{(1)}, x^{(2)}, \cdots, x^{(n)}, \cdots$. We assume that for any arc of the curve (x), the above sequence has a limit point x^* as n tends to infinity. Under these conditions, L. BIRAN has shown that the sequences, obtained by means of all those points taken on an arc of the curve (x), also admit x^* as a limit point [1].

Instead of using the sequences formed by the centers of spherical curvatures belonging to an arc of a space curve, P. VINCENSINI [2] used a sequence of points $x^{(1)}, x^{(2)}, \cdots, x^{(n)}, \cdots$ which is defined in any manner. Let θ_n denote the angle between the tangents to the curves $(x^{(n-1)})$ and $(x^{(n)})$ at the point $x^{(n-1)}$ and $x^{(n)}$ respectively and suppose that $\lim_{n\to\infty} \theta_n \neq 0$. If we take s as the arc lengths of the curves considered, the tangent vectors at the points $x^{(n-1)}$ and $x^{(n)}$ will be the velocity vectors. The limit point x of the sequence $x^{(1)}, x^{(2)}, \cdots, x^{(n)}, \cdots$, will be a function of s and if it reduces to a constant then its velocity will be zero. Since we assumed the existence of the limit of $x^{(n)}$ and denoted it by x^*; the velocity vector at x^* is the limiting position of the velocity vectors at the points $x^{(n-1)}$ and $x^{(n)}$. Let us suppose that the limit of θ_n (θ_n now being the angle between the two velocity vectors) is not zero. Then the two velocity vectors have the same limit if and only if they both vanish and for this reason x^* reduces to a constant. Therefore under the following two conditions 1) $\lim_{n\to\infty} x^{(n)}$ exists and 2) $\lim_{n\to\infty} \theta_n \neq 0$ the sequence of points $x^{(1)}, x^{(2)}, \cdots, x^{(n)}, \cdots$ has a fixed limit point and this limit point remains fixed while the point x traces the curve (x) or a given arc of (x).

2. Now, instead of taking the space curve mentioned above, let us consider the spherical curve $\mathbf{x}(s)$, $(\mathbf{x}'^2 = 1)$ which is drawn on a unit sphere and let the unit vectors

(1) $$\mathbf{b}_1 = \mathbf{x} \quad , \quad \mathbf{b}_2 = \mathbf{x}' \quad , \quad \mathbf{b}_3 = \mathbf{b}_1 \wedge \mathbf{b}_2$$

be the unit vectors of its DARBOUX trihedron. As a sequence of points corresponding to a point x on the curve, we shall take the sequence determined by the point having the position vector $\mathbf{x}^{(1)}$ which is given by

$$\mathbf{x}^{(1)} = \frac{\sigma \mathbf{b}_1 + \mathbf{b}_3}{\sqrt{1+\sigma^2}} \; ; \tag{2}$$

this point will be called the center of the spherical curvature at a point x on the curve. The scalar function σ which appears in (2) denotes the spherical curvature of (x) at x.

As the point x traces out the spherical curve (x), generally the point $x^{(1)}$ traces out another spherical curve $(x^{(1)})$ having as spherical trihedron the trihedron defined by the formulae

$$\left\{\begin{array}{l} \mathbf{b}_1^{(1)} = \mathbf{x}^{(1)} = \dfrac{\sigma \mathbf{b}_1 + \mathbf{b}_3}{\sqrt{1+\sigma^2}} \, , \\[2ex] \mathbf{b}_2^{(1)} = \mathbf{x}^{(1)\prime} = \dfrac{\mathbf{b}_1 - \sigma \mathbf{b}_3}{\sqrt{1+\sigma^2}} \, , \quad \dfrac{ds}{ds_1} = \dfrac{1+\sigma^2}{\sigma'} \, , \\[2ex] \mathbf{b}_3^{(1)} = \mathbf{b}_1^{(1)} \wedge \mathbf{b}_2^{(1)} = \mathbf{b}_2 \, . \end{array}\right. \tag{3}$$

If we apply the same process for $(x^{(1)})$ we obtain a new curve $(x^{(2)})$ and continuing in the same manner we get a sequence consisting of spherical curves. The two successive curves $(x^{(n-1)})$ and $(x^{(n)})$ satisfy the following relations,

$$\left\{\begin{array}{l} \mathbf{b}_1^{(n)} = \dfrac{\sigma_{n-1} \mathbf{b}_1^{(n-1)} + \mathbf{b}_3^{(n-1)}}{\sqrt{1+\sigma_{n-1}^2}} \, , \\[2ex] \mathbf{b}_2^{(n)} = \dfrac{\mathbf{b}_1^{(n-1)} - \sigma_{n-1} \mathbf{b}_3^{(n-1)}}{\sqrt{1+\sigma_{n-1}^2}} \, , \quad \dfrac{ds_{n-1}}{ds_n} = \dfrac{1+\sigma_{n-1}^2}{\sigma'_{n-1}} \, , \\[2ex] \mathbf{b}_3^{(n)} = \mathbf{b}_1^{(n)} \wedge \mathbf{b}_2^{(n)} = \mathbf{b}_2^{(n-1)} \, , \end{array}\right. \tag{4}$$

here σ_{n-1} denotes the spherical curvature of the curve $(x^{(n-1)})$; s_{n-1} and s_n denote the arc lengths of corresponding curves ([3], p. 21).

It can easily be seen from the relation (4) that the tangent vector $\mathbf{b}_2^{(n)}$ to the curve $(x^{(n)})$, is perpendiculer to the tangent vector $\mathbf{b}_2^{(n-1)}$ to $(x^{(n-1)})$ and this property does not depend on n. Hence, the limit of θ_n (where θ_n is the angle between the two successive curves) is equal to $\frac{\pi}{2}$ as n tends to infinity and obviously it is different from zero. For this reason, according to P. VINCENSINI's theorem, if the sequence of points $x^{(1)}, x^{(2)}, \cdots, x^{(n)}, \cdots$ which is formed by the centers of the spherical curvatures, has a limit, then this limit is the same for the curve (or for an arc of (x)) (x).

3. The above results, obtained for a spherical curve drawn on a unit sphere, can be extended to ruled surfaces.

In three dimensional Euclidean space a straight line X is completely determined by giving the vectors $\mathbf{x}$ and $\mathbf{x}_0$ where $\mathbf{x}$ and $\mathbf{x}_0$ denote the direction and moment vectors (having the same initial point O) respectively, ($\mathbf{x}^2 = 1$, $\mathbf{x} \cdot \mathbf{x}_0 = 0$). The dual vector $\mathbf{X}$ which is defined by the relation

$$\mathbf{X} = \mathbf{x} + \varepsilon \, \mathbf{x}_0 \quad , \quad (\varepsilon^2 = 0) \tag{5}$$

is a unit dual vector since $\mathbf{X}^2 = \mathbf{x}^2 = 1$. Thus to every directed line X there corresponds a unit dual vector $\mathbf{X}$ defined by (5) and also, to every straight line of the E_3 space there corresponds the dual point $\mathbf{X}$ on the dual unit sphere, which will be called Σ.

Since, a ruled surface is one that can be generated by the motion of a straight line

$$\mathbf{X}(s) = \mathbf{x}(s) + \varepsilon\, \mathbf{x}_0(s) \tag{6}$$

with s as a parameter, it follows that to a ruled surface (X) there corresponds a dual curve on Σ. We can choose s, which appears in (6), as the arc lenghth of the line of striction of the ruled surface (X).

It is known that the trihedron assigned to the generator X_1 of the ruled surface (X), is defined by the relations

$$\mathbf{X} = \mathbf{X}_1 \quad , \quad \mathbf{X}_2 = \frac{\mathbf{X}'}{P}\left(\mathbf{X}' = \frac{d\mathbf{X}(s)}{ds}\ , \ P = \sqrt{\mathbf{X}'^2}\right)\ , \quad \mathbf{X}_3 = \mathbf{X}_1 \wedge \mathbf{X}_2 \tag{7}$$

here, $\mathbf{X}_1$ is the direction of the generator of the ruled surface (X); the vector $\mathbf{X}_2$ is the normal to the surface at the central point M on $\mathbf{X}_1$ and $\mathbf{X}_3$ is the vector which is perpendicular to both $\mathbf{X}_1$ and $\mathbf{X}_2$ at the point M. $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is called BLASCHKE's trihedron for the line X_1 on the ruled surface (X). By virtue of $\mathbf{X}_i = \mathbf{x}_i + \varepsilon\, \mathbf{x}_{i0}$, $(i = 1, 2, 3)$ the following formulae of BLASCHKE

$$\left\{\begin{aligned} \mathbf{X}_1' &= P\,\mathbf{X}_2 \\ \mathbf{X}_2' &= -P\,\mathbf{X}_1 + Q\,\mathbf{X}_3 \\ \mathbf{X}_3' &= -Q\,\mathbf{X}_2 \end{aligned}\right. \qquad \left\{\begin{aligned} P &= \sqrt{\mathbf{X}'^2} = p + \varepsilon\, p_0 \\ Q &= \frac{(\mathbf{X}, \mathbf{X}', \mathbf{X}'')}{\mathbf{X}'^2} = q + \varepsilon\, q_0 \end{aligned}\right. \tag{8}$$

$$\left\{\begin{aligned} \mathbf{x}_1' &= p\,\mathbf{x}_2 \\ \mathbf{x}_2' &= -p\,\mathbf{x}_1 + q\,\mathbf{x}_3 \\ \mathbf{x}_3' &= -q\,\mathbf{x}_2 \end{aligned}\right. \qquad \left\{\begin{aligned} \mathbf{x}_{10}' &= p_0\,\mathbf{x}_2 + p\,\mathbf{x}_{20} \\ \mathbf{x}_{20}' &= -p_0\,\mathbf{x}_1 + q_0\,\mathbf{x}_3 - p\,\mathbf{x}_{10} + q\,\mathbf{x}_{30} \\ \mathbf{x}_{30}' &= -q_0\,\mathbf{x}_2 - q\,\mathbf{x}_{20} \end{aligned}\right. \tag{8'}$$

are satisfied where P and Q are the dual curvature and torsion respectively of (X) ([1], p. 334-5).

4. Any straight line $X^{(1)}$ can be expressed in terms of the vectors of BLASCHKE's trihedron $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ associated to the generator X_1 on the ruled surface (X), in the form

$$\mathbf{X}^{(1)} = B_1\,\mathbf{X}_1 + B_2\,\mathbf{X}_2 + B_3\,\mathbf{X}_3\,, \tag{9}$$

the quantities B_1, B_2, B_3, appearing in (9) are dual quantities such that

$$B_1^2 + B_2^2 + B_3^2 = 1 \tag{10}$$

and $B_i = b_i + \varepsilon\, b_{i0}$. Let $X^{(1)}$ be the instantaneous screwing axis of the trihedron; by using the conditions B_i = constant, $\dfrac{d\mathbf{X}^{(1)}}{ds} = 0$ in an analogous way to (2), we obtain

$$\mathbf{X}^{(1)} = \frac{Q\,\mathbf{X}_1 + P\,\mathbf{X}_3}{\sqrt{P^2 + Q^2}} \tag{11}$$

([8], p. 3). Hence, the instantaneous screwing axis $X^{(1)}$ of BLASCHKE's trihedron is perpendicular to X_2.

Now, we shall act as we have done in § 2. The instantaneous screwing axis of BLASCHKE's trihedron belonging to the ruled surface (X), generally, generates a ruled surface $(X^{(1)})$ which admits the trihedron

$$(12)\quad \begin{cases} \mathbf{X}_1^{(1)} = \mathbf{X}^{(1)} = \dfrac{Q\,\mathbf{X}_1 + P\,\mathbf{X}_3}{\sqrt{P^2 + Q^2}} \\[2ex] \mathbf{X}_2^{(1)} = \dfrac{\mathbf{X}^{(1)\prime}}{P^{(1)}} = \left[\dfrac{P\,\mathbf{X}_1 - Q\,\mathbf{X}_3}{(P^2 + Q^2)^{3/2}} \, (PQ' - QP') \right] \dfrac{1}{P^{(1)}} \dfrac{ds}{ds_1} \\[2ex] \mathbf{X}_3^{(1)} = \mathbf{X}_1^{(1)} \wedge \mathbf{X}_2^{(1)} = \mathbf{X}_2 \end{cases}$$

as BLASCHKE's trihedron. By continuing in this manner we obtain the sequence $(X^{(1)})$, $(X^{(2)}), \cdots, (X^{(n)}), \cdots$ of ruled surfaces and the trihedrons of BLASCHKE corresponding to two successive ruled surfaces $(X^{(n-1)})$ and $(X^{(n)})$ satisfy (just as the formulae (4)) the following relations :

$$(13)\quad \begin{cases} \mathbf{X}_1^{(n)} = \dfrac{Q^{(n-1)}\,\mathbf{X}_1^{(n-1)} + P^{(n-1)}\,\mathbf{X}_3^{(n-1)}}{\sqrt{(P^{(n-1)})^2 + (Q^{(n-1)})^2}} \\[2ex] \mathbf{X}_2^{(n)} = \left[\dfrac{P^{(n-1)}\,\mathbf{X}_1^{(n-1)} - Q^{(n-1)}\,\mathbf{X}_3^{(n-1)}}{[\,(P^{(n-1)})^2 + (Q^{(n-1)})^2\,]^{3/2}} \, (P^{(n-1)}\,Q^{(n-1)\prime} - Q^{(n-1)}\,P^{(n-1)\prime}) \right] \cdot \\ \qquad\qquad \cdot \dfrac{1}{P^{(n)}} \dfrac{ds_{n-1}}{ds_n} \\[2ex] \mathbf{X}_3^{(n)} = \mathbf{X}_2^{(n-1)} \end{cases}$$

As we can see from the above formulae $\mathbf{X}_1^{(n)}$ and $\mathbf{X}_2^{(n)}$ are perpendicular to $\mathbf{X}_2^{(n-1)}$ and the vector $\mathbf{X}_3^{(n)}$ is parallel to the vector $\mathbf{X}_2^{(n-1)}$. ([8], p. 6).

Let us consider the instantaneous screwing axes $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ introduced above. Thus we get a sequence $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ of straight lines for a generator X of the ruled surface (X); and the sequences $(\mathbf{X}_i^{(1)}), (\mathbf{X}_i^{(2)}), \cdots, (\mathbf{X}_i^{(n)}), \cdots, (i = 1, 2, 3)$ corresponding to $(\mathbf{X}_i)$ (BLASCHKE's trihedron) with $i = 1, 2, 3$ belonging to a generator X of the ruled surface (X) and finally we obtain a sequence $(X^{(1)}), (X^{(2)}), \cdots, (X^{(n)}), \cdots$ which corresponds to the ruled surface (X). Now, let us assume that the sequence of generators $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ has a limit as n tends to infinity. According to this, the limit of the two instantaneous screwing axes which are the generators of the two successive ruled surfaces $(X^{(n-1)})$ and $(X^{(n)})$ will be the same as the limits of the two trihedrons of BLASCHKE $(\mathbf{X}_i^{(n-1)})$ and $(\mathbf{X}_i^{(n)})$ connected with those generators, whereas the normals to these two successive surfaces are perpendicular. On the other hand, since the following relations hold

$$\mathbf{X}_2^{(n)} = \frac{\mathbf{X}^{(n)\prime}}{P^{(n)}} \quad , \quad \mathbf{X}_2^{(n-1)} = \frac{\mathbf{X}^{(n-1)\prime}}{P^{(n-1)}}$$

the limits of the vectors $\mathbf{X}^{(n-1)\prime}$ and $\mathbf{X}^{(n)\prime}$ are perpendicular and at the same time they are coincident ; then this limit vector being equal to zero, the limit of the sequence of the generators $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ is independent of s. For this reason, we see that the limit does not depend on the generator of the ruled surface (X); (or for some region on (X)), and therefore it is the same for all generators of the surface. Then, if the sequence $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ of straight lines has a limit X^*, this limit must be fixed.

5. Because of its close interest with the above, we now want to examine the infinitesimal elements of higher order of a ruled surface and an approximation of the n th order of a ruled surface. Let us consider the ruled surface (X) mentioned in § 4; and the sequence of successive surfaces $(X^{(1)}), (X^{(2)}), \cdots, (X^{(n)}), \cdots$ formed by the instantaneous screwing axes $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$ connected with the generator X_1 of the ruled surface (X). If we denote the dual curvature and torsion of the ruled surface (X), by $P = p + \varepsilon p_0$, $Q = q + \varepsilon q_0$ respectively and the dual angle of the lines X_1 and $X^{(1)}$ by $\Phi = \varphi + \varepsilon \varphi_0$, we find

$$\operatorname{tg} \Phi = \frac{P}{Q}$$

$$\operatorname{tg} \varphi = \frac{p}{q} \quad , \quad \varphi_0 = \frac{p_0 q - q_0 p}{p^2 + q^2}$$

In the above formulae φ is the real angle between the straight lines X_1 and $X^{(1)}$ and φ_0 is the shortest distance between these lines.

Indicating the dual curvature and the dual torsion of the ruled surface $(X^{(i)})$ by $P^{(i)}$ and $Q^{(i)}$ respectively and the dual angle between the straight lines $X^{(i)}$ and $X^{(i+1)}$ by $\Phi^{(i)}$, we have

$$(14) \qquad \operatorname{tg} \Phi^{(i)} = \operatorname{tg} (\varphi^{(i)} + \varepsilon \varphi_0^{(i)}) = \frac{P^{(i-1)} Q^{(i-1)\prime} - Q^{(i-1)} P^{(i-1)\prime}}{[(P^{(i-1)})^2 + (Q^{(i-1)})^2]^{3/2}}$$

[8]; where $\varphi^{(i)}$ and $\varphi_0^{(i)}$ denote, respectively, the angle and the shortest distance between $X^{(i)}$ and $X^{(i+1)}$. Since the central points $M^{(i)}$ and $M^{(i+1)}$ of the generators $X^{(i)}$ and $X^{(i+1)}$ are the feet of the common perpendicular to the generators, wc thus find

$$\varphi_0^{(i)} = M^{(i)} M^{(i+1)}$$

We may take the arc length s of the line of striction of a ruled surface (X), as the real parameter on which the dual curvature and dual torsion of (X), depend. If θ is the angle between the generator X_1 and the tangent to the line of striction we have

$$(15) \qquad p_0 = \sin \theta \quad , \quad q_0 = \cos \theta$$

If $P(s)$ and $Q(s)$ are given it is known that the ruled surface (X) is completely determined except for a displacement. Since $P(s) = p(s) + \varepsilon p_0(s)$ and $Q(s) = q(s) + \varepsilon q_0(s)$, the choice of the dual quantities P and Q are equivalent with the choice of three functions of a real variable, namely:

$$p = p(s) \quad , \quad q = q(s) \quad , \quad \theta = \theta(s)$$

Therefore these three functions determine the ruled surface (X) except for a displacement.

We, now consider the two ruled surfaces $(X^{(i-1)})$ and $(X^{(i)})$ which are determined by the two successive instantaneous screwing axes $X^{(i-1)}$ and $X^{(i)}$. Let $M^{(i-1)}$ and $M^{(i)}$ be the successive central points; we shall denote the successive angles between the corresponding generators and the tangents to the lines of striction by $\theta^{(i-1)}$ and $\theta^{(i)}$. Then, since $\mathbf{X}_3^{(i)} = \mathbf{X}_2^{(i-1)}$, by (13), the line of striction of the ruled surface $(X^{(i)})$ is given by the equation

$$\mathbf{M}^{(i)} = \mathbf{M}^{(i-1)} + \varphi_0^{(i-1)} \mathbf{X}_2^{(i-1)} .$$

Differentiating the above relation with respect to s and making use of BLASCHKE's formulae (8), we have

$$\mathbf{t}^{(i)} ds_i = [\, \mathbf{t}^{(i-1)} + \varphi^{(i-1)\prime} \mathbf{x}_2^{(i-1)} + \varphi_0^{(i-1)} (-p^{(i-1)} \mathbf{x}_1^{(i-1)} + q^{(i-1)} \mathbf{x}_3^{(i-1)}) \,] \, ds_{i-1}$$

where s_{i-1} and s_i are the arc lengths of the lines of striction of the ruled surfaces $(X^{(i-1)})$ and $(X^{(i)})$; $\mathbf{t}^{(i-1)}$ and $\mathbf{t}^{(i)}$ are used to indicate the unit tangents to these lines of striction.

On substituting the values

$$\mathbf{t}^{(i)} = \mathbf{x}_1^{(i)} \cos \theta^{(i)} + \mathbf{x}_3^{(i)} \sin \theta^{(i)}$$

$$\mathbf{x}_1^{(i)} = \mathbf{x}_1^{(i-1)} \cos \theta^{(i-1)} + \mathbf{x}_3^{(i-1)} \sin \theta^{(i-1)} \quad , \quad \mathbf{x}_3^{(i)} = \mathbf{x}_2^{(i-1)}$$

for $\mathbf{t}^{(i)}$, $\mathbf{x}_1^{(i)}$ and $\mathbf{x}_3^{(i)}$ in the above relation we get

$$\text{(16)} \qquad \operatorname{tg} \theta^{(i)} = \frac{\varphi_0^{(i-1)\prime} \cos \varphi^{(i-1)}}{\cos \theta^{(i-1)} - \varphi_0^{(i-1)} p^{(i-1)}} .$$

The dual relation (14), states that the angle $\varphi^{(i)}$ and the shortest distance $\varphi_0^{(i)}$ of the successive instantaneous axes $X^{(i-1)}$ and $X^{(i)}$, are functions of p, q, θ and of their first $(i-1)$ derivatives. That is to say

$$\text{(17)} \quad \begin{cases} \varphi^{(i)} = f_1 \left(p, q, \theta, \dfrac{dp}{ds}, \dfrac{dq}{ds}, \dfrac{d\theta}{ds}, \ldots, \dfrac{d^{(i-1)}p}{ds^{(i-1)}}, \dfrac{d^{(i-1)}q}{ds^{(i-1)}}, \dfrac{d^{(i-1)}\theta}{ds^{(i-1)}} \right) \\[2ex] \varphi_0^{(i)} = f_2 \left(p, q, \theta, \dfrac{dp}{ds}, \dfrac{dq}{ds}, \dfrac{d\theta}{ds}, \ldots, \dfrac{d^{(i-1)}p}{ds^{(i-1)}}, \dfrac{d^{(i-1)}q}{ds^{(i-1)}}, \dfrac{d^{(i-1)}\theta}{ds^{(i-1)}} \right) \end{cases}$$

Moreover, by virtue of the relations (16) and (17) we see that the angle $\theta^{(i)}$, (between the tangent to the line of striction of the surface $(X^{(i)})$ and the straight line $X^{(i)}$) depends on p, q, θ and their first $(i-1)$ derivatives, namely

$$\text{(18)} \qquad \theta^{(i)} = f_3 \left(p, q, \theta, \frac{dp}{ds}, \frac{dq}{ds}, \frac{d\theta}{ds}, \ldots, \frac{d^{(i-1)}p}{ds^{(i-1)}}, \frac{d^{(i-1)}q}{ds^{(i-1)}}, \frac{d^{(i-1)}\theta}{ds^{(i-1)}} \right)$$

$$(i = 1, 2, \ldots, (n+1))$$

Thus, if we consider the $(n+1)$ angles $\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(n+1)}$; the $(n+1)$ distances $\varphi_0^{(1)}, \varphi_0^{(2)}, \ldots, \varphi_0^{(n+1)}$ and the angles $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n+1)}$ which are defined by the $(n+2)$ instantaneous screwing axes $X^{(1)}, X^{(2)}, \ldots, X^{(n+2)}$, we obtain the relations (17) and (18) which form a system of $3n+3$ equations in the $3n+3$ unknowns

$$\text{(19)} \qquad \begin{cases} p, \dfrac{dp}{ds}, \dfrac{d^2 p}{ds^2}, \ldots, \dfrac{d^n p}{ds^n} \\[2ex] q, \dfrac{dq}{ds}, \dfrac{d^2 q}{ds^2}, \ldots, \dfrac{d^n q}{ds^n} \\[2ex] \theta, \dfrac{d\theta}{ds}, \dfrac{d^2 \theta}{ds^2}, \ldots, \dfrac{d^n \theta}{ds^n} . \end{cases}$$

Therefore the set of successive instantaneous screwing axes $X^{(1)}, X^{(2)}, \dots, X^{(n+2)}$ and the set of angles $\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(n+1)}$ give a representation of the natural properties of a ruled surface in the neighbourhood of a generator X_1.

6. In order obtain an approximation of order n of a plane curve (x) in the neighbourhood of one of its points, we may consider the center of curvature $x^{(1)}$ of the curve (x) at a point x, the center of curvature $x^{(2)}$ of the evolute of (x) at the point $x^{(1)}$ and so forth: by continuing the same process, we find the set of points $x^{(1)}, x^{(2)}, \dots, x^{(n)}$. The radius of curvature $\varrho_n = \overline{x^{(n)}\, x^{(n+1)}}$ of the evolute $(x^{(n)})$ depends on the radius of curvature ϱ (of the curve (x)) and its first n derivatives. Let (x_n) be the circle of radius ϱ_n and having its center at the point $x^{(n+1)}$. Taking the evolute of the circle (x_n) which passes through the point $x^{(n-1)}$ and denoting it by (x_{n-1}) and applying the same operation on the circle (x_{n-1}) passing through the point $x^{(n-2)}$, we obtain the set of curves $(x_n), (x_{n-1}), \dots, (x_1)$. The curve (x_1) passes through the point x and satisfies the $(n+1)$ equations formed by $\varrho, \varrho', \varrho'', \dots, \varrho^{(n)}$ and the distances $\overline{xx^{(1)}}, \overline{x^{(1)}x^{(2)}}, \dots, \overline{x^{(n)}\, x^{(n+1)}}$ assigned to (x). We may define (x_1), as an approximation of the n th order of (x) in the neighbourhood of a point x [5].

The above properties may easily be extended to ruled surfaces. Let (X) be the ruled surface generated by the motion of the straight line X_1. As X_1 generates the ruled surface (X), the instantaneous screwing axes of BLASCHKE's trihedron $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$, generates the surface (X^*) which we will be called the evolute of the ruled surface (X); the involutes of the ruled surface (X^*) are all the ruled surfaces (X) which admit X^* as an evolute. [6]

Let us take into consideration the ruled surface (X) generated by the straight line X_1 and denote the natural elements of X_1 by p, q, and θ. Consider the successive instantaneous screwing axes $X^{(1)}, X^{(2)}, \dots, X^{(n+2)}$ and let $p^{(n+1)}, q^{(n+1)}, \theta^{(n+1)}$ be the elements belonging to the generator $X_1^{(n+1)}$ of the ruled surface $(X^{(n+1)})$. The ruled helicoid (X_{n+1}) having the elements $p^{(n+1)}, q^{(n+1)}, \theta^{(n+1)}$ and the axes $X^{(n+2)}$, admits $X^{(n+1)}$ as a generator.

Let us choose an involute of (X_{n+1}) which contains the straight line $X^{(n)}$. Likewise, taking the involute of the ruled surface (X_n), containing the straight line $X^{(n-1)}$ and continuing in the same manner we obtain the sequence of ruled surafaces, that is to say (X_n), $(X_{n-1}), \dots, (X_1)$. If (X_1) is the last term of the above sequence, it contains the generator X_1 of the ruled surface (X). *It will be noted that the quantities* $\varphi^{(i)}, \varphi_0^{(i)}, \theta^{(i)}$ *are the same for the ruled surfaces* $(X^{(i)})$ *and* (X_i).

Therefore the quantities belonging to the ruled surfaces X and X_1 which are given by (19), satisfy a system of $3n+3$ equations in the $3n+3$ quantities:

$$\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(n+1)}$$

$$\varphi_0^{(1)}, \varphi_0^{(2)}, \dots, \varphi_0^{(n+1)}$$

$$\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(n+1)}$$

We may define the ruled surface (X_1), as an approximation of the n th order of the ruled surface (X) in the neighbourhood of the generator X_1 of (X).

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ÖZET

Bu çalışmada esas itibariyle, regle yüzeye bağlı üçyüzlülerin ânî vidalanma eksenlerinin teşkil ettiği doğru dizileri ; ayrıca, çalışma ile çok yakın ilgisi dolasiyle, bir regle yüzeyin yüksek mertebeden infinitezimal elemanları ve bir ana doğrusu cıvarında *n*. mertebeden yaklaşımı incelenmiştir.