HYPER-ASYMPTOTIC CURVES OF A RIEMANNIAN HYPERSURFACE [1]

S. C. RASTOGI

The hyper-asymptotic curves of a hypersurface belonging to a Riemannian space are those curves of the hypersurface such that at each point of the curves a particular geodesic hypersurface of the Riemannian space, analogue to the rectifying plane of a curve in ordinary Eulidean space, contains an element of a given vector field associated to the hypersurface, [1], [2]. The differential equation of hyper-asymptotic curves is obtained by a new method and some properties of these curves are shown.

1. Introduction. A hyper-asymptotic curve in an Euclidean-space of three dimensions was defined by MISHRA [1], [2] [2]), who also studied the hyper-asymptotic curves in a hypersurface of a Riemannian space [3]. A hyper-asymptotic curve is a curve on a surface which has the property that its rectifying plane at all points contains a line of a specified rectilinear congruence through that point. In the present paper we shall obtain the differential equation of hyper-asymptotic curves in a hypersurface of a Riemannian space in a manner different from the method of MISHRA [3]. We shall also study them in relation to geodesics and asymptotic curves. The expression for the hyper-asymptotic curvature in a hypersurface of a Riemannian space is also obtained.

2. Vector fields in V_n. Let x^i $(i = 1, \ldots, n)$ be the coordinates of a point on a hypersurface V_n which is embedded in a Riemannian space V_{n+1} for which the coordinates of a point are given by y^α $(\alpha = 1, \ldots, n + 1)$. For points in V_n the fundamental matrix $\|\partial y^\alpha/\partial x^i\|$ is of rank n. Let the metrics of V_n and V_{n+1}, which are supposed to be positive definite, be denoted by

$$\Phi = g_{ij}\, dx^i\, dx^j, \qquad (i, j = 1, \ldots, n) \tag{2.1}$$

and

$$\Psi = a_{\alpha\beta}\, dy^\alpha\, dy^\beta, \qquad (\alpha, \beta = 1, \ldots, n + 1)\ ^{3)}, \tag{2.2}$$

respectively. Then the metric tensors of V_n and V_{n+1} are related as follows:

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j} = g_{ij}, \tag{2.3}$$

where $y^\alpha{}_{,i}$ denotes the covariant differentiation of y^α with respect to x^i.

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2) Numbers in square brackets refer to the references at the end of this paper.

3) In what follows Greek indices take the values (1,..., n).

If N^α are the components of a unit vector in V_{n+1} normal to V_n, then

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, N^\beta = 0, \tag{2.4}$$

and

$$a_{\alpha\beta}\, N^\alpha N^\beta = 1. \tag{2.5}$$

If a vector field in V_n has components U^α in the y's and components U^i in the x's, then U^α and u^i are related as follows :

$$U^\alpha = y_{\alpha,i}\, u^i. \tag{2.6}$$

Let λ^α be contravariant components in V_{n+1} of a unit vector in the direction of a geodesic curve of a congruence: then we must have

$$\lambda^\alpha = y^\alpha{}_{,i}\, t^i + r\, N^\alpha, \tag{2.7}$$

where t^i are the components of a vector in the hypersurface and r is a parameter. If θ be the angle between the vectors whose contravariant components are λ^α and N^α, then from equation (2.7) we obtain

$$\cos\theta = a_{\alpha\beta}\, \lambda^\alpha N^\beta = r. \tag{2.8}$$

With the help of (2.3), (2.4), (2.5) and (2.7) we have

$$a_{\alpha\beta}\, \lambda^\alpha \lambda^\beta = 1 = g_{ij}\, t^i t^j + r^2. \tag{2.9}$$

From (2.8) and (2.9) we get

$$\sin^2\theta = g_{ij}\, t^i t^j. \tag{2.10}$$

Since we know that a curve in V_n, which is immersed in V_{n+1}, can be regarded as a curve either in V_{n+1} or in V_n and hence has n curvatures and n normals relative to V_n. We shall denote curvatures and unit normals relative to V_{n+1} of a curve in V_n, by K_r $(r = 1, \ldots, n)$ and $\eta_r|^\alpha (r = 2, \ldots, n+1)$ and those relative to V_n by K_r' $(r = 1, \ldots, n-1)$ and $\xi_r|^i (r = 2, \ldots, n)$ respectively. The unit tangent to the curve will be denoted by $\eta_1|^\alpha$ or $\xi_1|^i$ according as it is regarded as a vector in V_{n+1} or in V_n respectively.

3. Hyper-asymptotic curves. As an analogue to the rectifying plane in ordinary space a geodesic-surface in V_{n+1} is introduced. It is determined by the tangent vector to the curve C with equations $x^i = x^i(s)$ in V_n (s denoting arc length) and by the second normal of C in V_{n+1}. If the vector with contravariant components λ^α, lies in this geodesic-surface, it can be expressed as a linear combination of $\eta_1|^\alpha$ and $\eta_3|^\alpha$. Hence we have

$$\lambda^\alpha = a\, \eta_1|^\alpha + c\, \eta_3|^\alpha, \tag{3.1}$$

where a and c are to be determined.

Comparing (2.7) with (3.1) we have

$$y^\alpha{}_{,i}\, t^i + r\, N^\alpha = a\, \eta_1|^\alpha + c\eta_3|^\alpha. \tag{3.2}$$

Differentiating the relation $\eta_2|^\alpha = \xi_2|^i\, y^\alpha{}_{,i}$ with respect to the arc length s of the curve we obtain

(3.3) $$K_2\,\eta_3|^\alpha = K_2'\,\xi_3|^i\, y^\alpha{}_{,i} + \Omega_{ij}\,\xi_1|^i\,\xi_2|^j\, N^\alpha.$$

Writing $K_n|_2$ in place of $\Omega_{ij}\,\xi_1|^i\,\xi_2|^j$, equation (3.2) takes the form

$$y^\alpha{}_{,i}\, t^i + r\,N^\alpha = a\,\eta_1|^\alpha + \frac{c}{K_2}\,(K_2'\,\xi_3|^i\, y^\alpha{}_{,i} + K_n|_2\, N^\alpha),$$

or

(3.4) $$y^\alpha{}_{,i}\, t^i + r\,N^\alpha = a\,\eta_1|^\alpha + b\,(K_2'\,\xi_3|^i\, y^\alpha{}_{,i} + K_n|_2\, N^\alpha),$$

where

$$b = \frac{c}{K_2}.$$

On multiplying (3.4) by $a_{\alpha\beta}\,N^\beta$ and summing on α and making use of (2.4) and (2.5) we have

(3.5) $$r = b\,K_n|_2,$$

since $$a_{\alpha\beta}\,N^\beta\,\eta_1|^\alpha = 0.$$

Again multiplying equations (3.4) by $a_{\alpha\beta}\,y^\beta{}_{,j}$ and summing on α, we have by the use of equation (2.3) and (2.4) the n equations

(3.6) $$g_{ij}\, t^i = a\, g_{ij}\,\xi_1|^i + b\,K_2'\, g_{ij}\,\xi_3|^i,$$

where we have used $\eta_1|^\alpha = y^\alpha{}_{,i}\,\xi_1|^i$, as in (2.6).

Now by multiplying equation (3.6) by $\xi_1|^j$ and summing on j and using the results $g_{ij}\,\xi_1|^i\,\xi_1{}^j = 1$ and $g_{ij}\,\xi_1|^j\,\xi_3|^i = 0$, we obtain

(3.7) $$g_{ij}\, t^i\,\xi_1|^j = a.$$

Substituting for a and b from (3.5) and (3.7) in (3.6) we get

(3.8) $$g_{ij}\, t^i = g_{ij}\,\xi_1|^i\,(g_{pq}\, t^p\,\xi_1|^q) + \frac{r}{K_n|_2}\, K_2'\, g_{ij}\,\xi_3|^i,$$

multiplication of equation (3.8) by g^{ik}, summation on j and the replacement of t^k/r by l^k leads to equation

(3.9) $$K_2'\,\xi_3|^k - K_n|_2\,(l^k - g_{pq}\, l^p\,\xi_1|^k\,\xi_1|^q) = 0.$$

We shall call (3.9) the differential equation of hyper-asymptotic curves. We shall study these in the next section.

4. Some Properties. For a congruence specified by the parameters l^k, solutions of the n equations (3.9) determine the hyper-asymptotic curves in V_n relative to that congruence. The parameter r can not vanish under the assumption that the direction λ^α is not in V_n. The left hand members of equations (3.9) may be denoted by Γ^k, which we shall call the contravariant components of the hyper-asymptotic curvature vector. A hyper-asymptotic curve of V_n with respect to a congruence determined by the parameters l^k may therefore be defined as a curve along which the hyper-asymptotic curvature vector is a null vector.

Using the relation $g_{ij}\,\xi_1|^i\,\xi_1|^j = 1$, the equation (3.9) can be written in the form

$$\Gamma^k \equiv K_2'\,\xi_3|^k - K_n|_2\,\nu^k = 0, \tag{4.1}$$

where

$$\nu^k = g_{ij}\,\xi_1|^i\,(l^k\,\xi_1|^j - l^j\,\xi_1|^k). \tag{4.2}$$

In (4.1) if $K_n|_2 = 0$ (i.e. C is an asymptotic curve of order 2, HAYDEN [4]) and if $\Gamma^k = 0$; we get either $K_2' = 0$, (i.e., C is *a* geodesic of order 2, SRIVASTAVA [5]) or the vector with components $\xi_3|^k$ is a null vector.

Hence we have the following :

Theorem (4.1) . *The necessary and sufficient condition for a hyper-asymptotic curve to be an asymptotic curve of order* 2 *is given by either of the following* :

(i) *it be a geodesic of order* 2.

(ii) *the vector with components* $\xi_3|^k$ *be a null vector.*

In addition to this, if $K_2' = 0$ and if $\Gamma^k = 0$ we get either $K_n|_2 = 0$ or the vector with components ν^k is a null vector. Thus we have the following :

Theorem (4.2) . *The necessary and sufficient condition for a hyper-asymptotic curve to be a geodesic of order* 2, *is given by either of the following* :

(i) *it be an asymptotic curve of order* 2.

(ii) *the vector with components* ν^k *be a null vector.*

The magnitude K_h of the vector Γ is given by

$$K_n^2 = g_{ij}\,\Gamma^i\,\Gamma^j. \tag{4.3}$$

Since $t^k/r = l^k$ then with the help of (2.8) and (2.10) we obtain the following equation

$$g_{ij}\,l^i\,l^j = \tan^2\theta. \tag{4.4}$$

The angle α between the vector l^k and $\xi_1|^k$ is given by

$$\cos\alpha = g_{ij}\,l^i\,\xi_1|^k. \tag{4.5}$$

From (4.3), (4.1), (4.4) and (4.5) we obtain that in terms of θ and α, the magnitude of the hyper-asymptotic curvature vector is given by

$$K_n = K'_2 - K_n|_2\tan\theta\sin\alpha, \tag{4.6}$$

where K_2' is the curvature of the curve which is the geodesic of order 2 in V_n and $K_n|_2$ is the curvature of the curve which is an asymptotic curve of order 2. It is observed that if $\theta = 0$, the hyper-asymptotic curve is a geodesic of second order.

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DEPARMENT OF MATHEMATICS
LUCKNOW UNIVERSITY
LUCKNOW, (INDIA).

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ÖZET

Bir Riemann uzayına ait bir hiperyüzey üzerindeki hiperasimptotik eğriler, hiperyüzey üzerine çizilmiş öyle eğrilerdir ki bunların her noktasında Riemann uzayının özel bir geodezik hiperyüzeyi, ilk verilen hiperyüzey üzerinde tanımlanan bir vektör alanının o noktadaki elamanını ihtiva eder ; yukarda söz konusu edilen geodezik hiperyüzey, Euklid uzaymdaki bir eğrinin rektifyan düzleminin bir bakımdan benzeri kabul edilebilir [1], [2]. Bu araştırmada, hiperasimptotik eğrilerin diferansiyel denklemi yeni bir yoldan elde edilmiş ve bu eğrilerin bâzı özellikleri gösterilmiştir.