A PROBLEM IN CONGRUENCES OF CURVES IN A RIEMANNIAN SPACE

M. D. UPADHYAY

In this paper a definition of a «Generalised normal congruence» has been given and necessary and sufficient conditions that a congruence be generalised normal have been obtained. The conditions determining the «Ultimate Hypersurfaces» of such a congruence have also been derived.

1. Introduction. Congruences of curves in a Riemannian space imbedded in V_m $(m > n)$ have been studied by UPADHYAY [2], [3] [1]) and others. In the present work we shall define *Generalised normal congruences* and *Ultimate Hypersurfaces* and obtain the conditions determining them.

Let the fundamental forms of V_n and V_m be positive definite and be given by $g_{ij}\, dx^i\, dx^j$ and $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$ respectively. Then we have [2])

$$g_{ij} = a_{\alpha\beta}\, y^\alpha_{;i}\, y^\beta_{;j}\,, \quad (i, j, k, \ldots = 1, 2, \ldots, n), \quad (\alpha, \beta = 1, 2, \ldots, m) \tag{1.1}$$

where a semi colon (;) followed by a Latin index denotes tensor derivative with respect to the x's.

We consider a set of $m-n$ congruences of curves in V_m, such that one curve of each congruence passes through each point of the subspace V_n. Let $s_{\tau|}$ $(\tau = n+1, \ldots, m)$ be the length of a curve of a congruence measured from the point $P\,(x^i)$ at which the curve intersects V_n to another point Q on the curve. V_n is known as the subspace of reference.

2. Generalised normal congruence. A normal congruence in a Riemannian space is one whose curves intersect orthogonally a family of hypersurfaces. We define a *Generalised normal congruence* in V_m as the congruence whose curves intersect orthogonally a family of subspaces of the type V_n given by the relation

$$\Phi = \Phi\,(x^1, x^2, \ldots x^n) = \text{Constant.} \tag{2.1}$$

When a congruence is normal to a family of subspaces, the gradient of Φ at each point is parallel to the vector $(\partial\, y^\alpha_{\tau|} / \partial\, s_{\tau|})$, where $y^\alpha_{\tau|}$ are the coordinates of a point on the curve of the congruence, τ being fixed.

Let $\left(\partial\, y^\alpha_{\tau|} / \partial\, s_{\tau|}\right)_i$ be the covariant components of the vector $\left(\partial\, y^\alpha_{\tau|} / \partial s_{\tau|}\right)$. Then from the definition given above, the condition for the congruence to be normal is

1) The numbers in square brackets denote the corresponding references at the end of the paper.

2) Latin letters $i, j, k, \ldots$ take values from 1 to n; early Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ have the range from 1 to m and later letters $\mu, \nu, \tau, \ldots$ range from $n+1$ to m.

$$\frac{\Phi_{,1}}{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_1} = \frac{\Phi_{,2}}{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_2} = \cdots = \frac{\Phi_{,n}}{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_n} = \Gamma, \tag{2.2}$$

where Γ is the point function and comma (,) followed by an index denotes covariant derivative of Φ with respect to x^i, say. Thus (2.2) is equivalent to the expression

$$\Phi_{,i} = \Gamma\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_i. \tag{2.3}$$

The condition that such a function Φ may exist is given by LEVI-CIVITA [1]

$$\begin{aligned} &\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_i \left\{\frac{\partial}{\partial x^k}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_j - \frac{\partial}{\partial x^j}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_k\right\} \\ &+ \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_j \left\{\frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_k - \frac{\partial}{\partial x^k}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_i\right\} \\ &+ \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_k \left\{\frac{\partial}{\partial x^j}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_i - \frac{\partial}{\partial x^i}\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_j\right\}, \end{aligned} \tag{2.4}$$

or

$$\begin{aligned} &\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_i \left\{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{j,k} - \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{k,j}\right\} \\ &+ \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_j \left\{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{k,i} - \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{i,k}\right\} \\ &+ \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_k \left\{\left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{i,j} - \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\right)_{j,i}\right\}. \end{aligned} \tag{2.5}$$

Thus, (2.5) *is the necessary and sufficient condition that the congruence-$\lambda_{\tau|}$ be generalised normal.*

3. Particular case. When $m = n + 1$, the congruence is such that its curves intersect orthogonally a family of hypersurfaces given by the relation

$$\Phi(x^1, x^2, \ldots, x^n) = \text{Constant}. \tag{3.1}$$

The condition that the congruence be normal takes the following form

$$\begin{aligned} &\left(\frac{\partial y^{\alpha}}{\partial s}\right)_i \left\{\left(\frac{\partial y^{\alpha}}{\partial s}\right)_{j,k} - \left(\frac{\partial y^{\alpha}}{\partial s}\right)_{k,j}\right\} \\ &+ \left(\frac{\partial y^{\alpha}}{\partial s}\right)_j \left\{\left(\frac{\partial y^{\alpha}}{\partial s}\right)_{k,i} - \left(\frac{\partial y^{\alpha}}{\partial s}\right)_{i,k}\right\} \end{aligned} \tag{3.2}$$

$$+\left(\frac{\partial y^{\alpha}}{\partial s}\right)_{k}\left\{\left(\frac{\partial y^{\alpha}}{\partial s}\right)_{i,j}-\left(\frac{\partial y^{\alpha}}{\partial s}\right)_{j,i}\right\}.$$

Thus (3.2) is the necessary and sufficient condition that the single congruence of curves with one curve through each point of V_n, be normal.

4. Ultimate hypersurfaces. Let M and M' be adjacent curves of the congruence - $\lambda_{\tau|}$ and let the contravariant components of their position vectors be $y^{\alpha}_{\tau|}$ and $y^{\alpha}_{\tau|} + y^{\alpha}_{\tau|\,;\,i}\, dx^i + \frac{\partial^2 y^{\alpha}_{\tau|}}{\partial s_{\tau|}^2}\, ds_{\tau|}$ respectively. Let the unit tangents at M and M' be $(\partial y^{\alpha}_{\tau|}/\partial s_{\tau|})$ and $\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}} + \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}^2}\right)_{,i} dx^i + \frac{\partial^2 y^{\alpha}_{\tau|}}{\partial s_{\tau|}^2}\, ds_{\tau|}$ respectively. Since MM' is perpendicular to the tangent at M, we have

$$a_{\alpha\beta}\,\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\cdot\left(y^{\beta}_{\tau|\,;\,i}\, dx^i + \frac{\partial y^{\beta}_{\tau|}}{\partial s_{\tau|}}\, ds_{\tau|}\right) = 0.$$

UPADHYAY [2] has obtained the following

(4.1) $$ds_{\tau|} = -\,\bar{p}_{\tau|\,;}\, dx^i$$

where

(4.2) $$p_{\tau|\,;} \stackrel{\text{def}}{=} a_{\alpha\beta}\,\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, y^{\beta}_{\tau|,\,j}\,.$$

The contravariant components of the vector along MM' is

$$y^{\alpha}_{\tau|\,;\,i}\, dx^i + \frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, ds_{\tau|}$$

which by virtue of (4.1) takes the form

(4.3) $$y^{\alpha}_{\tau|\,;\,i}\, dx^i - p_{\tau|\,i}\,\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}}\, dx^i \stackrel{\text{def}}{=} U^{\alpha}_{\tau|}\,.$$

The moment of the vector whose contravariant components $U^{\alpha}_{\tau|}$ is given by the determinant

(4.4) $$|\, E_{\tau|\,i\,j}\, dx^i\, dx^j \,| \stackrel{\text{def}}{=} \left[\, U^{\alpha}_{\tau|}\;\; V^{\beta}_{\tau|}\;\; T^{\gamma}_{\tau|}\,\right].$$

where

$$V^{\beta}_{\tau|} \stackrel{\text{def}}{=} \left(\frac{\partial y^{\beta}_{\alpha|}}{\partial s_{\tau|}}\right)_{,j} dx^j - p_{\tau|\,j}\,\frac{\partial^2 y^{\beta}_{\alpha|}}{\partial s_{\tau|}^2}\, dx^j$$

$$T^{\gamma}_{\tau|} \stackrel{\text{def}}{=} \frac{\partial y^{\gamma}_{\tau|}}{\partial s_{\tau|}}\,.$$

(4.4) can be written as

$$\left[U^{\alpha}_{\tau|} \; V^{\beta}_{\tau} \; T^{\gamma}_{\tau|} \right] = [B_{\tau|ij} + C_{\tau|ij}] \, dx^i \, dx^j$$

where

(4.5) a $$E_{\tau|ij} = B_{\tau|ij} = C_{\tau|ij},$$

(4.5) b $$B_{\tau|ij} \overset{\text{def}}{=} \text{Det.} \left[y^{\alpha}_{\tau|ij} \; \left(\frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}} \right)_{,j} \; \frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}} \right],$$

(4.5) c $$C_{\tau|ij} \overset{\text{def}}{=} \text{Det.} \left[y^{\alpha}_{\tau|\,;\,i} \; \frac{\partial^2 y^{\alpha}_{\tau|}}{\partial s_{\tau|}^2} \; \frac{\partial y^{\alpha}_{\tau|}}{\partial s_{\tau|}} \right] p_{\tau|j}$$

If MM' be of zero moment, we have

(4.6) $$| \, E_{\tau|ij} \, dx^i \, dx^j \, | = 0$$

In analogy with the definition of ultimate surfaces in a Euclidean space of three dimensions WEATHERBURN [4, 225] ultimate hypersurfaces are defined as the hypersurfaces for which all the directions determined by the ratio $dx^i : dx^j$ $(i \neq j)$ in (4.6) are coincident. Thus ultimate hypersurfaces will be determined by

(4.7) $$\text{Det.} \; | \, E_{\tau|ij} \, | = 0.$$

REFERENCES

[1] LEVI CIVITA T. : The Absolute Differential Calculus (Translated by Miss M. LONG) BLACKIE AND SON, London, (1927)

[2] UPADHYAY M. D. : *Congruences of curves in a Riemannian space* - I, İstanbul Üniv. Fen Fak. Mec. Ser. A, 27, 10-40 (1962).

[3] UPADHYAY M. D. : *Congruences of curves in a Riemannian space* - II, İstanbul Üniv. Fen Fak. Mec. Ser. A, 19, 19-22 (1954).

[4] WEATHERBURN C. E. : Differential Geometry of three dimensions, II, CAMBRIDGE UNIVERSITY PRESS (1930)

DEPARTMENT OF MATHEMATICS
LUCKNOW UNIVERSITY
LUCKNOW, (INDIA).

(*Manuscript received February 2, 1969*)

ÖZET

Bu araştırmada m-boyutlu bir Riemann uzayında «genelleştirilmiş normal kangrüans» ların tanımı verilmiş ve bir kongrüansın genelleştirilmiş normal bir kongrüans olması için gerek ve yeter şart elde edilmiştir. Ayrıca, üç boyutlu Euclid uzayında göz önüne alınan kongrüansların «son yüzey» kavrāmı genelleştirilerek, «son hiperyüzey»leri belitten şartlar elde edilmiştir.