THE PERIODS OF OSCILLATION OF A ROTATING COLUMN OF LIQUID IN THE PRESENCE OF AN AXIAL MAGNETIC FIELD AND A UNIFORM AXIAL CURRENT

K. M. SRIVASTAVA AND R. C. SHARMA

The periods of oscillation of a rotating column of a perfectly conducting liquid in the presence of an axial magnetic field and a uniform axial current are derived. It is found that the presence of an axial magnetic field and a uniform axial volume current splits each characteristic value (in the absence of the magnetic field and axial volume current) into two. A criterion for the frequencies to be real has also been obtained.

I. Introduction. LORD KELVIN (1910) has first investigated the problem of the periods of oscillation of a rigidly rotating column of an inviscid liquid. FULTZ (1959) has demonstrated in his experiments the excitation of axisymmetric modes by means of a small disc on the axis of a rotating cylinder containing water. CHANDRASEKHAR in his treatise (1961) [3, 390] has discussed the problem of the periods of oscillation of a rotating column of liquid in the presence of a uniform axial magnetic field. GUPTA (1968) has studied the problem of the periods of oscilllation of a rotating column of liquid in the presence of a uniform axial volume current.

The object of the present note is to study the periods of oscillation of a rotating column of a perfectly conducting liquid in the presence of an axial magnetic field and a uniform axial volume current.

II. Formulation of the problem and solutions. We consider the stationary circular flows of an incompressible, inviscid and perfectly conducting fluid between two rotating coaxial cylinders in the presence of an axial magnetic field and a uniform axial volume current. The non-dissipative equations of hydromagnetics [3, 332, for vanishing ν and η], then, allow the stationary solution

$$
\begin{aligned}
&u_r = u_z = 0 \quad , \quad u_\theta = V(r) = r\,\Omega(r), \\
&H_r = 0 \quad , \quad H_0 = \Omega' r \,, H_z = H = \text{constant},
\end{aligned} \tag{1}
$$

where Ω and Ω' are constant.

We then consider an infinitesimal perturbation of the flow represented by the solution (1) and take the perturbed physical variables as

$$
u_r \,,\; V + u_\theta \,,\; u_z \,,\; h_r \,,\; H_\theta(r) + h_\theta \,,\; H + h_z \,,\; \widetilde{\omega} = \delta\pi. \tag{2}
$$

where

$$
\frac{d\pi}{dr} = -\frac{V^2}{r} + \frac{\mu_e}{4\pi\varrho}\cdot\frac{H_\theta^2}{r}\,. \tag{3}
$$

The linearized equations governing these perturbations are

(4) $$\frac{\partial u_r}{\partial t} + \Omega \frac{\partial u_r}{\partial \theta} - 2\Omega u_\theta - \frac{\mu_e}{4\pi\varrho}\left[\frac{H_\theta}{r}\frac{\partial h_r}{\partial \theta} - 2\frac{H_0}{r} h_\theta + H\frac{\partial h_r}{\partial z}\right] = -\frac{\partial \widetilde{\omega}}{\partial r},$$

(5) $$\frac{\partial u_r}{\partial t} + \Omega \frac{\partial u_0}{\partial \theta} + \left(\frac{dV}{dr} + \frac{V}{r}\right) u_r - \frac{\mu_e}{4\pi\varrho}\left[\frac{H_\theta}{r}\frac{\partial h_0}{\partial \theta} + 2\frac{H_\theta}{r} h_r + H\frac{\partial h_\theta}{\partial z}\right] = -\frac{1}{r}\frac{\partial \widetilde{\omega}}{\partial \theta},$$

(6) $$\frac{\partial u_z}{\partial t} + \Omega \frac{\partial u_z}{\partial \theta} - \frac{\mu_e}{4\pi\varrho}\left(H\frac{\partial h_z}{\partial z} + \frac{H_\theta}{r}\frac{\partial h_z}{\partial 0}\right) = -\frac{\partial \widetilde{\omega}}{\partial z},$$

(7) $$\frac{\partial h_r}{\partial t} + \Omega \frac{\partial h_r}{\partial \theta} - H\frac{\partial u_r}{\partial z} - \frac{H_\theta}{r}\frac{\partial u_r}{\partial \theta} = 0,$$

(8) $$\frac{\partial h_\theta}{\partial t} + \Omega \frac{\partial h_0}{\partial \theta} - H\frac{\partial u_\theta}{\partial z} - \frac{H\theta}{r}\frac{\partial u_\theta}{\partial \theta} = 0,$$

(9) $$\frac{\partial h_z}{\partial t} + \Omega \frac{\partial h_z}{\partial \theta} - \frac{H_0}{r}\frac{\partial u_z}{\partial \theta} - H\frac{\partial u_z}{\partial z} = 0,$$

(10) $$\frac{\partial ur}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0,$$

and

(11) $$\frac{\partial h_r}{\partial r} + \frac{h_r}{r} + \frac{1}{r}\frac{\partial h_\theta}{\partial \theta} + \frac{\partial h_z}{\partial z} = 0.$$

Analysing the disturbance into normal modes, we seek solutions of the above equations whose dependence on t, θ and z is given by

(12) $$e^{i(pt+m\theta+kz)},$$

where p is a constant (which can be complex), m is an integer (positive, zero, or negative) and k is the wave number of the disturbance in the z-direction.

Let $u_r(r)$, $u_\theta(r)$, $u_z(r)$, $h_r(r)$, $h_0(r)$, $h_z(r)$ and $\widetilde{\omega}(r)$ now denote the amplitudes of the various perturbations whose (t, θ, z) dependence is given by (12). Equations (4) - (11) then give

(13) $$i\sigma u_r - 2\Omega u_\theta - \frac{\mu_e}{4\pi\varrho}(im\Omega' h_r - 2\Omega' h_\theta + ikH\, h_r) = -D\widetilde{\omega},$$

(14) $$i\sigma u_0 + 2\Omega u_r - \frac{\mu_e}{4\pi\varrho}(im\Omega' h_\theta + 2\Omega' h_r + ikH\, h_\theta) = -\frac{im}{r}\widetilde{\omega}$$

(15) $$i\sigma u_z - \frac{\mu_e}{4\pi\varrho}(im\Omega' h_z + ikH\, h_z) = -ik\widetilde{\omega},$$

(16) $$i\sigma h_r = i(kH + m\Omega')u_r,$$

(17) $$i\sigma h_\theta = i(kH + m\Omega')u_\theta,$$

(18) $$i\sigma h_z = i(kH + m\Omega')u_z,$$

(19) $$Du_r + \frac{u_r}{r} + \frac{im}{r} u_\theta + ik\, u_z = 0\,,$$

and

(20) $$Dh_r + \frac{h_r}{r} + \frac{im}{r} h_\theta + ik\, h_z = 0\,,$$

where

(21) $$\sigma = p + m\Omega \qquad \text{and} \qquad D = \frac{d}{dr}\,.$$

Introducing the Lagrangian displacement $\vec{\xi}$ defined by

(22) $$u_r = i\sigma\xi_r\,,\ u_\theta = i\sigma\xi_\theta\,,\ u_z = i\sigma\xi_z\,,$$

and inserting for $\vec{u}$ in terms of $\vec{\xi}$ in equations (16) - (18), we get

(23) $$h_r = i(kH + m\Omega')\xi_r\,,\ h_\theta = i(kH + m\Omega')\xi_\theta\,,\ \ h_z = i(kH + m\Omega')\xi_z\,.$$

Now substituting for $\vec{u}$ and $\vec{h}$ in accordance with equations (22) and (23) in equation (13) -(15), we find after some reduction

(24) $$(\sigma^2 - \Omega_H^2)\xi_r + 2i\left(\sigma\Omega - \frac{\Omega_H^2}{kH + m\Omega'}\right)\xi_\theta = D\widetilde{\omega}\,,$$

(25) $$(\sigma^2 - \Omega_H^2)\,\xi_\theta - 2i\left(\sigma\Omega - \frac{\Omega_H^2\Omega'}{kH + m\Omega'}\right)\xi_r = \frac{im}{r}\widetilde{\omega}\,,$$

(26) $$(\sigma^2 - \Omega_H^2)\,\xi_z = ik\,\widetilde{\omega}\,,$$

where

(27) $$\Omega_H^2 = \frac{\mu_e}{4\pi\varrho}\,(kH + m\Omega')^2\,.$$

Again using equations, (22) and (23), both the equations of continuity (19) and the solenoidal equations (20) (from $\Delta.\ H = 0$) reduce to

(28) $$D\xi_r + \frac{\xi_r}{r} + \frac{im}{r}\xi_\theta + ik\,\xi_z = 0\,.$$

Multiplying equations (25) and (26) by $-\frac{im}{r}$ and $-ik$, respectively, adding and making use of (28), we obtain

(29) $$(\sigma^2 - \Omega_H^2)\left(D\xi_r + \frac{\xi_r}{r}\right) - \frac{2m}{r}\left(\sigma\Omega - \frac{\Omega_H^2\Omega'}{kH + m\Omega'}\right)\xi_r = \left(\frac{m^2}{r^2} + k^2\right)\widetilde{\omega}\,;$$

while eliminating ξ_θ between equations (24) and (25), we obtain

(30)
$$\left[\sigma^2 - \Omega_H^2 - \frac{4\left(\sigma\Omega - \dfrac{\Omega_H^2\Omega'}{kH+m\Omega'}\right)^2}{(\sigma^2-\Omega_H^2)}\right]\xi_r$$

$$= D\tilde{\omega} + \frac{2m}{r}\cdot\frac{\left(\sigma\Omega - \dfrac{\Omega_H^2\Omega'}{kH+m\Omega'}\right)\tilde{\omega}}{(\sigma^2-\Omega_H^2)}.$$

Since the radial velocity vanishes at the boundary of the column of radius R, it follows from (22) that

(31)
$$\xi_r = 0 \quad \text{at} \quad r = R.$$

This condition is consistent with the assumption of a rigid boundary at $r = R$, as in the experiments of FULTZ.

Eliminating ξ_r between (29) and (30), we obtain

(32)
$$\frac{d^2\tilde{\omega}}{dr^2} + \frac{1}{r}\frac{d\tilde{\omega}}{dr} + \left(\beta^2 - \frac{m^2}{r^2}\right)\tilde{\omega} = 0,$$

where

(33)
$$\beta^2 = \frac{4k^2\left[\sigma\Omega - \dfrac{\Omega_H^2\Omega'}{kH+m\Omega'}\right]^2}{(\sigma^2-\Omega_H^2)} - k^2.$$

The boundary condition (32) with the help of (30) and (31) reduces to

(34)
$$D\tilde{\omega} + \frac{2m}{r}\cdot\frac{\left(\sigma\Omega - \dfrac{\Omega_H^2\Omega'}{kH+m\Omega'}\right)}{(\sigma^2-\Omega_H^2)}\tilde{\omega} = 0, \quad \text{at } n = R.$$

The solution of (32) regular at the origin is

(35)
$$\tilde{\omega} = C\, J_m\,(\beta r),$$

C being a constant and J_m being the BESSEL function of order m. Applying the boundary condition (34), we then obtain

(36)
$$\alpha\, J_m'(\alpha) + \frac{2m\left(\sigma\Omega - \dfrac{\Omega_H^2\Omega'}{kH+m\Omega'}\right)}{(\sigma^2-\Omega_H^2)} J_m(\alpha) = 0,$$

with $\alpha = \beta R$.

Comparing (32) and (34) with the corresponding equation in the absenec of axial magnetic field and the axial current [8, chapter VII, equations 88, 89] we find that the equations become identical if σ in the nonmagnetic case is replaced by

$$\Omega(\sigma^2 - \Omega_H^2)\Big/\left(\sigma\Omega - \frac{\Omega_H^2\Omega'}{kH+m\Omega'}\right).$$

Thus if σ_0 be the characteristic value for a given wave number of the hydrodynamic problem, the characteristic value σ for the present hydromagnetic problem is given by

(37)
$$\sigma_0 = \frac{\Omega(\sigma^2 - \Omega_H^2)}{\left(\sigma\Omega - \frac{\Omega_H^2\Omega'}{kH + m\Omega'}\right)} .$$

Thus, the presence of an axial magnetic field and a uniform axial volume current splits each characteristic value σ (in the absence of the magnetic field and axial current) into two, σ_1 and σ_2 such that

(38)
$$\sigma_{1,2} = \frac{\sigma_0\Omega \pm \sqrt{(\sigma_0\Omega)^2 - 4\Omega\left[\frac{\Omega_H^2\sigma_0\Omega'}{kH + m\Omega'} - \Omega\Omega_H^2\right]}}{2\,\Omega}$$

From equation (38), we infer that the frequencies are real if $\Omega > \sigma_0\Omega'/(kH + m\Omega')$. The con- for the frequencies to be real, thus depends on the strength of the axial magnetic field as well as the axial volume current. It has been shown by GUPTA [4, 71] that in the presence of a uniform axial volume current, the characteristic frequency is split into two and that the frequencies are always real if $\Omega > (\mu_e/4\pi\varrho)^{1/2}\,\Omega'$.

For the axisymmetric mode $m = 0$, it is clear from equation (38) that the frequencies are real, depending on the strength of the axial magnetic field.

REFERENCES

[1] LORD KELVIN, T.W. : Mathematical and Physical papers iv, Hydrodynamics and General Dynamics, CAMBRIDGE, 152, (1910).

[2] FULTZ, D. J. Meteorology, 16, 199, (1959).

[3] CHANDRASEKHAR, S. : Hydrodynamic and Hydromagnetic Stability, OXFORD UNIVERSITY PRESS, London, (1961).

[4] GUPTA, A. S. : Progress of Mathematics, 2, I, 68, (1968).

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF JODHPUR,
JODHPUR, INDIA.

(Manuscipt received May 5 th, 1969.)

ÖZET

Eksenin doğrultusuna göre yönelmiş bir manyetik alan ve yine eksenin doğrultusunda geçirilen bir düzgün elektrik akımın etkisi altında bulunan, ekseni etrafında dönen lam iletken bir sıvı akımın salınımlarının peryodları elde edilmiştir. Bu eksensel manyetik alanın ve düzgün ceryanının varlığı (manyetik alan ve eksensel hacım akımın yokluğu halinde) her has değeri iki kısma ayırdığı gösterilmektedir. Frekansların gerçel olmalarını sağlayan bir kriter verilmiştir.