ON HYPER DARBOUX LINES IN FINSLER SUBSPACE

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The hyper DARBOUX lines (hyper D-lines) on a surface of a 3-dimensional Euclidean space have been defined and studied by PRVANOVITCH [2]. SINGH [5] investigated some properties of these curves for a Riemannian subspace. The object of this paper is to obtain the differential equation of the hyper D-lines of a FINSLER subspace and deduce some properties of union hyper D-lines.

1. Fundamental formulae. Let a subspace F_m given by the equations $x^i = x^i(u^\alpha)$ $(i = 1, \dots, n; \alpha = 1, \dots, m)$ be immersed in an n-dimensional FINSLER space F_n. Consider any arbitrary curve $C : u^\alpha = u^\alpha(s)$ (or $x^i = x^i(s)$) of the subspace. The components x'^i $\left(\text{or } \dfrac{dx^i}{ds}\right)$ and u'^α $\left(\text{or } \dfrac{du^\alpha}{ds}\right)$ of the unit tangent vector of C are related by $x'^i = B^i_\alpha\, u'^\alpha$ where $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$.

The metric tensors $g_{ij}(x,x')$ and $g_{\alpha\beta}(u,u')$ of F_n and F_m are such that

$$g_{\alpha\beta}(u,u') = g_{ij}(x,x')\, B^i_\alpha B^j_\beta \tag{1.1}$$

and the tensor

$$C_{ijk}(x,x') = \frac{1}{2}\, \frac{\partial g_{ij}(x,x')}{\partial x'^k} \tag{1.2}$$

satisfies the identities

$$C_{ijk}(x,x')\, x'^i = C_{ijk}(x,x')\, x'^j = C_{ijk}(x,x')\, x'^k = 0. \tag{1.3}$$

There exist $(n-m)$ vectors, called secondary normal vectors $\overset{*}{n}{}^i_{(\mu)}$ $(\mu = m+1, \dots, n)$, which satisfy the relations

(1.4) a) $$g_{ij}(x,x')\, \overset{*}{n}{}^j_{(\mu)} B^i_\alpha = \overset{*}{n}_{(\mu)i}\, B^i_\alpha = 0,$$

b) $$g_{ij}(x,\overset{*}{n}_{(\mu)})\, \overset{*}{n}{}^i_{(\mu)}\, \overset{*}{n}{}^j_{(\mu)} = 1,$$

c) $$g_{ij}(x,x')\, \overset{*}{n}{}^i_{(\mu)}\, \overset{*}{n}{}^j_{(\nu)} = \delta^\mu_\nu\, \psi_{(\mu)} \text{ (no summation on } \mu).$$

The δ-derivative of a vector field V^i of F_n along the curve C has been defined by RUND [8] as

$$\frac{\delta V^i}{\delta s} = \frac{dV^i}{ds} + p^{*i}_{jk} V^j \frac{dx^k}{ds} \tag{1.5}$$

where p^{*i}_{jk} are the connection coefficients of the embedding space.

The δ-derivative of the metric tensor $g_{ij}(x,x')$ along the curve C is given by RUND [8]

$$\frac{\delta}{\delta s} g_{ij}(x,x') = 2C_{ijk}(x,x') \frac{\delta x'^k}{ds} = C^*_{ijk} \frac{dx^k}{ds} \tag{1.6}$$

where

$$C^*_{ijk}(x,x') = g_{ij;k}(x,x'). \tag{1.7}$$

If $q^i \left(= \frac{\delta x'^i}{\delta s} \right)$ and $p^\alpha \left(= \frac{\delta u'^\alpha}{\delta s} \right)$ are the components of the first curvature vectors of the curve with respect to F_n and F_m respectively, we have

$$q^i = p^\alpha B^i_\alpha + \sum_\mu \Omega^*_{(\mu)\alpha\beta} \ (u,u') \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \eta^{*i}_{(\mu)} \tag{1.8}$$

where $\Omega^*_{(\mu)\alpha\beta}$ are the components of the secondary second fundamental tensors of the subspace.

2. **Frenet's formulae.** After putting $n^i_{(0)} = \frac{dx^i}{ds}$ and $n^i_{(1)}$ as a unit vector along q^i, we have the following equations

$$q^i = \frac{\delta n_{(0)}}{\delta s} = K_{(1)} \ n_{(1)} \tag{2.1}$$

where $K_{(1)}$ is the first curvature of the curve. The vector $n^i_{(1)}$ is consistent with the conditions

$$g_{ij}(x,x') n^i_{(0)} \ n^j_{(1)} = 0 \text{ and } g_{ij} \ (x,x') n^i_{(0)} = 1. \tag{2.2}$$

The δ - derivative {refer (1b.5)} of the vector $n^i_{(1)}$ gives

$$\frac{\delta}{\delta s} \ n^i_{(1)} = \frac{d}{ds} n^i_{(1)} + P^{*i}_{jk} \ n^j_{(1)} \frac{dx^k}{ds} \cdot \tag{2.3}$$

Decomposing this vector in the surface determined by $n^i_{(0)}$ and $n^i_{(0)}$ and orthogonal to this surface we have

$$\frac{\delta}{\delta s} n^i_{(1)} = a \ n^i_{(0)} + b \ n^i_{(1)} + c \ n^i_{(2)} \tag{2.4}$$

where the vector $n^i_{(2)}$ satisfies

(2.5) $$g_{ij}(x,x')\, n^i_{(p)}\, n^j_{(2)} = \delta^p_2 \quad (p = 0, 1, 2).$$

A simple calculation based on the equations (1.3), (1.6) (2.1), (2.4) and (2.5) gives

$$a = -K_{(1)}\,;\; b = -\frac{1}{2}\, C^*_{ijk} \frac{dx^k}{ds}\, n^i_{(1)}\, n^j_{(1)}\,.$$

Substituting the values of a, b in (2.4) and putting $C = K_{(2)}$ (the second curvature of the curve) we get

(2.6) $$\frac{\delta}{\delta s}\, n^i_{(1)} = -K_{(1)}\, n^i_{(0)} + K\, n^i_{(2)} - \frac{1}{2}\, C^*_{ljk} \frac{dx^k}{ds}\, n^l_{(1)}\, n^j_{(1)}\, n^i_{(1)}\,.$$

The equations (2.1) and (2.6) represents the first two FRENET's formulae for a curve in the FINSLER space.

3. **Hyper D-line.** Consider a congruence of curves given by the vector field λ^i such that through each point of F_m there passes exactly one curve of the congruence. At the point of the subspace we have

(3.1) $$\lambda^i = t^{*\alpha} B^i_\alpha + \sum_\mu C^*_{(\mu)}\, n^{*i}_{(\mu)}\,.$$

The curve C is said to be a hyper D-line of the subspace if the surface spanned by the vectors $n^i_{(0)}$ and $R_{(1)}\, n^i_{(1)} + R_{(2)} \frac{dR_{(1)}}{ds}\, n^i_{(2)}$ contains the vector λ^i. ($R_{(1)} = \frac{1}{K_{(1)}}$ and $R_{(2)} = \frac{1}{K_{(2)}}$ are the radii of curvatures of the first and second order respectively.) From (2.1) and (2.6) we have

(3.2) $$\frac{\delta^2}{\delta s^2}\, n^i_{(0)} = \frac{dK_{(1)}}{ds}\, n^i_{(1)} - K^2_{(1)}\, n^i_{(0)} + K_{(1)}\, K_{(2)}\, n^i_{(2)} - \frac{1}{2}\, K_{(1)}\, C^*_{ljk} \frac{dx^k}{ds}\, n^l_{(1)}\, n^j_{(1)}\, n^i_{(1)}\,.$$

This equation gives

(3.3) $$g_{ij}(x,x') \left(\frac{\delta^2}{\delta s^2}\, n^i_{(0)}\right) \left(R_{(1)}\, n^j_{(1)} + R_{(2)}\, n^j_{(2)} \frac{dR_{(1)}}{ds}\right) = -\frac{1}{2}\, C^*_{ljk} \frac{dx^k}{ds}\, n^l_{(1)}\, n^j_{(1)}\,.$$

For the hyper D-liue, we have

(3.4) $$\lambda^i = A\left(R_{(1)}\, n^i_{(1)} + R_{(2)} \frac{dR_{(1)}}{ds}\, n^i_{(2)}\right) + B\, n^i_{(0)}\,.$$

Multiplying this equation by $g_{ij}(x,x')\, n^j_{(0)}$, $g_{ij}(x,x')\, n^j_{(1)}$ and $g_{ij}(x,x') \frac{\delta^2}{\delta s^2}\, n^j_{(0)}$ respectively and putting $\lambda_{(h)} = g_{ij}(x,x')\, \lambda^i\, n^j_{(h)}$ $(h = 0, 1, 2,)$ we get $B = \lambda_{(0)}$, $A = \lambda_{(1)}\, K_{(1)}$ and

(3.5) $$g_{ij}(x,x')\, \lambda^i \frac{\delta^2}{\delta s^2}\, n^j_{(0)} = -\frac{1}{2}\, \lambda_{(1)}\, K_{(1)}\, C^*_{ljk} \frac{dx^k}{ds}\, n^l_{(1)}\, n^j_{(1)}$$
$$+ \lambda_{(0)}\, g_{ij}(x,x')\, n^i_{(0)} \left(\frac{\delta^2}{\delta s^2}\, n^j_{(0)}\right).$$

which with the help of equation (3.2) gives

$$\frac{dK_{(1)}}{ds}\lambda_{(1)} + K_{(1)}K_{(2)}\lambda_{(2)} = 0. \tag{3.6}$$

This equation represents the equation of hyper D-line of the subspace.

Theorem (3.1). If the congruence λ^i is along the first curvature vector of the curve then the necessary and sufficient condition that it be a hyper D-line is that it be a curve of constant first curvature.

Proof. The proof follows from equation (3.6), the definition of $\lambda_{(1)}$ and $\lambda_{(2)}$ and the fact that the vectors $n^i_{(1)}$ and $n^i_{(2)}$ are orthogonal.

4. Hyper D-line envolving secondary second fundamental tensors. Writing the δ-differential of equation (1.8) and using [1]

$$\overset{*}{n}{}^i_{(\mu);\gamma} = -\psi_{(\mu)}B^i_\alpha\, g^{\alpha\delta}\,\overset{*}{\Omega}_{(\mu)\delta\gamma} - \overset{*}{C}_{jhk}\,g^{ji}B^k_\gamma\,\overset{*}{n}{}^h_{(\mu)} + \sum_\lambda N^{(\mu)}_{(\lambda)\gamma}\,\overset{*}{n}{}^i_{(\lambda)} \tag{4.1}$$

we obtain

$$\begin{aligned}\frac{\delta^2}{\delta s^2}n^i_{(0)} = {} & \Big[\frac{\delta p^\alpha}{\delta s} - \sum_\mu \psi_{(\mu)}\,\overset{*}{\Omega}_{(\mu)\theta\beta}\frac{du^\theta}{ds}\frac{du^\beta}{ds}\,\overset{*}{\Omega}_{(\mu)\delta\gamma}\,g^{\alpha\delta}\frac{du^\gamma}{ds}\Big]B^i_\alpha \\ & + \sum_\mu\Big[3\,\overset{*}{\Omega}_{(\mu)\alpha\beta}\,p^\alpha\frac{du^\beta}{ds} + \overset{*}{\Omega}_{(\mu)\alpha\beta;\gamma}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} \\ & + \sum_\lambda \overset{*}{\Omega}_{(\lambda)\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}N^{(\lambda)}_{(\mu)\gamma}\frac{du^\gamma}{ds}\Big]\overset{*}{n}{}^i_{(\mu)} \\ & - \sum_\mu \overset{*}{\Omega}_{(\mu)\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\,\overset{*}{C}_{lhk}\,g^{li}\,B^k_\gamma\,\overset{*}{n}{}^h_{(\mu)}\frac{du^\gamma}{ds}.\end{aligned} \tag{4.2}$$

Substituting the values of $\frac{\delta^2}{\delta s^2}n^i_{(0)}$ from (4.2) and λ^i from (3.1) in the equation (3.5) we get

$$(4.3)\qquad g_{\alpha\beta}(u,u')\frac{\delta p^{\alpha}}{\delta s}t^{*\beta}-\sum_{\mu}\psi_{(\mu)}K^{*}_{(\mu)}\Omega^{*}_{(\mu)\alpha\beta}\,t^{*\alpha}\frac{du^{\beta}}{ds}-\sum_{\mu}K^{*}_{(\mu)}\,C^{*}_{ihk}\,t^{*\varepsilon}\,B^{i}_{\varepsilon}\,n^{*h}_{(\mu)}\frac{dx^{k}}{ds}$$

$$+\sum_{\mu}\psi_{(\mu)}C^{*}_{(\mu)}\left[3\,\Omega^{*}_{(\mu)\alpha\beta}\,p^{\alpha}\frac{du^{\beta}}{ds}+\Omega^{*}_{(\mu)\alpha\beta;\gamma}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\frac{du^{\gamma}}{ds}+\sum_{\lambda}K^{*}_{(\lambda)}N^{(\lambda)}_{(\mu)\gamma}\frac{du^{\gamma}}{ds}\right]$$

$$-\sum_{\mu}\sum_{\nu}C^{*}_{(\nu)}\,K^{*}_{(\mu)}\,n^{*i}_{(\nu)}\,C^{*}_{ihk}\,n^{*h}_{(\mu)}\frac{dx^{k}}{ds}+\frac{1}{2}\lambda_{(1)}K_{(1)}\,C^{*}_{ljk}\frac{dx^{k}}{ds}\,n^{l}_{(1)}\,n^{j}_{(1)}$$

$$-\lambda_{(0)}\left[g_{\alpha\beta}\left[\frac{\delta p^{\alpha}}{\delta s}\frac{du^{\beta}}{ds}-\sum_{\mu}\psi_{(\mu)}K^{*2}_{(\mu)}\right]=0.\right.$$

where we have used

$$(4.4)\qquad K^{*}_{(\mu)}=\Omega^{*}_{(\mu)\alpha\beta}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}$$

and the fact that $n^{i}_{(0)}$, is orthogonal to $n^{*i}_{(\mu)}$.

The first two FRENET's formulae for the subspace are

$$(4.5)\qquad p^{\alpha}=\frac{\delta}{\delta s}\left(\frac{du^{\alpha}}{ds}\right)=k_{(1)}\,\xi^{\alpha}_{(1)},$$

$$(4.6)\qquad \frac{\delta\,\xi^{\alpha}_{(1)}}{\delta s}=-k_{(1)}\,\xi^{\alpha}_{(0)}+k_{(2)}\,\xi^{\alpha}_{(2)}-\frac{1}{2}\,C^{*}_{\beta\gamma\delta}\frac{du^{\delta}}{ds}\,\xi^{\alpha}_{(1)}\,\xi^{\gamma}_{(1)}\,\xi^{\beta}_{(1)}$$

where $C^{*}_{\beta\gamma\delta}=g_{\beta\gamma;\delta}$, $\xi^{\alpha}_{(0)}=\frac{du^{\alpha}}{ds}$, $\xi^{\alpha}_{(1)}$ and $\xi^{\alpha}_{(2)}$ are the first and second curvature vectors and $k_{(1)}$, $k_{(2)}$ are the first and second curvatures with respect to the subspace. These equations give

$$(4.7)\qquad \frac{\delta\,p^{\alpha}}{\delta s}=-k^{2}_{(1)}\,\xi^{\alpha}_{(0)}+\xi^{\alpha}_{(1)}\frac{dk_{(1)}}{ds}+k_{(1)}\,k_{(2)}\,\xi^{\alpha}_{(2)}-\frac{1}{2}\,k_{(1)}\,C^{*}_{\beta\gamma\delta}\frac{du^{\delta}}{ds}\,\xi^{\alpha}_{(1)}\,\xi^{\gamma}_{(1)}\,\xi^{\beta}_{(1)}\,.$$

Substituting the value of $\frac{\delta\,p^{\alpha}}{\delta s}$ in (4.3) and using the fact

$$\lambda_{(0)}=g_{ij}(x,x')\,\lambda^{i}\,n^{j}_{(0)}=g_{\alpha\beta}\;t^{*\alpha}\;\frac{du^{\beta}}{ds}=\;t^{*}_{(0)},$$

$$K_{(1)}\,\lambda_{(1)}=g_{ij}(x,x')\,\lambda^{\nu}\,q^{j}=g_{\alpha\beta}\,t^{*\alpha}\,p^{\beta}+\sum_{\mu}\psi_{(\mu)}\;C^{*}_{(\mu)}\;K^{*}_{(\mu)}$$

we obtain

$$(4.8)\qquad \frac{dk_{(1)}}{ds}\, \overset{*}{t}_{(1)} + k_{(1)}\, k_{(2)}\, \overset{*}{t}_{(2)} - \frac{1}{2}\, k_{(1)}\, C^{*}_{\beta\gamma\delta} \frac{du^{\delta}}{ds}\, \overset{*}{t}_{(1)}\, \xi^{\gamma}_{(1)}\, \xi^{\beta}_{(1)}$$

$$-\sum_{\mu} \psi_{(\mu)}\, K^{*}_{(\mu)}\, \Omega^{*}_{(\mu)\alpha\beta}\, t^{*\alpha} \frac{du^{\beta}}{ds} - \sum_{\mu} K^{*}_{(\mu)}\, C^{*}_{ihk}\, t^{*\varepsilon}\, B^{i}_{\varepsilon}\, n^{*h}_{(\mu)} \frac{dx^{k}}{ds}$$

$$+\sum_{\mu} \psi_{(\mu)}\, C^{*}_{(\mu)} \left[3\, \Omega^{*}_{(\mu)\alpha\beta}\, p^{\alpha} \frac{du^{\beta}}{ds} + \Omega^{*}_{(\mu)\alpha\beta\,;\,\gamma} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} + \sum_{\lambda} K^{*}_{(\lambda)}\, N^{(\lambda)}_{(\mu)\gamma} \frac{du^{\gamma}}{ds}\right]$$

$$-\sum_{\mu}\sum_{\nu} C^{*}_{(\nu)}\, K^{*}_{(\mu)}\, n^{*i}_{(\nu)}\, C^{*}_{ihk}\, n^{*h}_{(\mu)} \frac{dx^{k}}{ds} + \frac{1}{2}\, (k_{(1)}\, \overset{*}{t}_{(1)} + \sum_{\mu} \psi_{(\mu)}\, C^{*}_{(\mu)}\, K^{*}_{(\mu)}).$$

$$C^{*}_{ljk} \frac{dx^{k}}{ds}\, n^{l}_{(1)}\, n^{j}_{(1)} + \overset{*}{t}_{(0)} \sum_{\mu} \psi_{(\mu)}\, K^{*2}_{(\mu)} = 0$$

where

$$\overset{*}{t}_{(1)} = g_{\alpha\beta}\, t^{*\alpha}\, \xi^{\beta}_{(1)} \quad ; \quad \overset{*}{t}_{(2)} = g_{\alpha\beta}\, t^{*\alpha}\, \xi^{\beta}_{(2)}\,.$$

The equation (4.8) represents the hyper D-line of the subspace. The equation has been expressed in the secondary second fundamental tensors.

5. Hyper D-lines and union curves. The union curves of the subspace have been studied by SINGH [4]. For these curves we have

$$\lambda^{i} = A\, n^{i}_{(0)} + B\, \frac{\delta}{\delta s}\, n^{i}_{(0)}.$$

Using (3.1), (1.8) and (5.4), we obtain

$$t^{*\alpha} = A \frac{du^{\alpha}}{ds} + B\, p^{\alpha}\,,\ C^{*}_{(\mu)} = B\, \Omega^{*}_{(\mu)\alpha\beta} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds}\,,$$

$$\overset{*}{t}_{(0)} = A\,,\ \overset{*}{t}_{(1)} = B\, k_{(1)}\,,\ \overset{*}{t}_{(2)} = 0.$$

Substituting these values in (4.8) we get, after some simplification,

$$(5.1)\qquad k_{(1)} \frac{dk_{(1)}}{ds} + 2 \sum_{\mu} \psi_{(\mu)}\, K^{*}_{(\mu)}\, \Omega^{*}_{(\mu)\alpha\beta}\, p^{\alpha} \frac{du^{\beta}}{ds} + \sum_{\mu} \psi_{(\mu)}\, K^{*}_{(\mu)}\, \Omega^{*}_{(\mu)\alpha\beta;\gamma} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds}$$

$$+\sum_{\mu}\sum_{\gamma} \psi_{(\mu)}\, K^{*}_{(\mu)}\, K^{*}_{(\gamma)}\, N^{(\lambda)}_{(\gamma)\gamma} \frac{du^{\gamma}}{ds} - \sum K^{*}_{(\mu)}\, C^{*}_{ihk}\, p^{\varepsilon}\, B^{\varepsilon}_{i}\, n^{*h}_{(\mu)} \frac{ds^{k}}{ds}$$

$$-\sum_{\mu}\sum_{\nu} K^{*}_{(\mu)}\, K^{*}_{(\nu)}\, n^{*l}_{(\nu)}\, C^{*}_{ihk}\, n^{*h}_{(\mu)} \frac{dx^{k}}{ds} - \frac{1}{2}\, C^{*}_{\beta\gamma\delta} \frac{du^{\delta}}{ds}\, p^{\gamma}\, p^{\beta}$$

$$+\frac{1}{2}\, C^{*}_{ihk} \frac{dx^{k}}{ds}\, q^{i}\, q^{h} = 0\,.$$

Further since [1]

$$\psi_{(\gamma)} N^{(\mu)}_{(\lambda)\gamma} + \psi_{(\mu)} N^{(\lambda)}_{(\mu)\gamma} = \frac{\partial \psi_{(\mu)}}{\partial \mu^{\gamma}} \delta^{\lambda}_{\mu} - \overset{*}{C}_{ihk} \overset{*}{n}{}^{h}_{(\lambda)} \overset{*}{n}{}^{i}_{(\mu)} B^{k}_{\beta} \tag{5.2}$$

we have

$$\sum_{\lambda} \sum_{\mu} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\lambda)} \psi_{(\mu)} N^{(\lambda)}_{(\mu)\gamma} \frac{du^{\gamma}}{ds} = \frac{1}{2} \sum_{\mu} \overset{*}{K}{}^{2}_{(\mu)} \frac{d\psi_{(\mu)}}{ds} - \frac{1}{2} \sum_{\mu} \sum_{\lambda} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\lambda)} \overset{*}{C}_{ihk} \overset{*}{n}{}^{i}_{(\mu)} \overset{*}{n}_{(\lambda)} \frac{dx^{k}}{ds}. \tag{5.3}$$

A simple calculation based on the equations (1.1), (1.8) and (1.4) yields

$$\frac{1}{2} \overset{*}{C}_{ihk} \frac{dx^{k}}{ds} q^{i} q^{h} - \frac{1}{2} \overset{*}{C}_{\alpha\beta\gamma} \frac{du^{\gamma}}{ds} p^{\alpha} p^{\beta} = \frac{1}{2} \sum_{\mu} \sum_{\nu} \overset{*}{C}_{ihk} \frac{dx^{k}}{ds} \overset{*}{n}{}^{i}_{(\mu)} \overset{*}{n}{}^{h}_{(\nu)} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\nu)} + \sum_{\mu} \overset{*}{K}_{(\mu)} \overset{*}{C}_{ihk} \frac{dx^{k}}{ds} B^{h}_{\beta} p^{\beta} \overset{*}{n}{}^{i}_{(\gamma)}. \tag{5.4}$$

With the help of equations (5.3) and (5.4), the equation (5.1) reduces to

$$k_{(1)} \frac{dk_{(1)}}{ds} + 2 \sum_{\mu} \psi_{(\mu)} \overset{*}{K}_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta} p^{\alpha} \frac{du^{\beta}}{ds} + \sum_{\mu} \psi_{(\mu)} \overset{*}{K}_{(\mu)} \overset{*}{\Omega}_{(\mu)\alpha\beta;\gamma} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} + \frac{1}{2} \sum_{\mu} \overset{*}{K}{}^{2}_{(\mu)} \frac{d\psi_{(\mu)}}{ds} - \sum_{\mu} \sum_{\nu} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\nu)} \overset{*}{C}_{ihk} \frac{dx^{k}}{ds} \overset{*}{n}{}^{i}_{(\mu)} \overset{*}{n}{}^{h}_{(\nu)} = 0 \tag{5.5}$$

or

$$k_{(1)} \frac{dk_{(1)}}{ds} + \frac{1}{2} \frac{\delta}{\delta s} \left[\sum_{\mu} \psi_{(1)} \left(\overset{*}{\Omega}_{(\mu)\alpha\beta} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \right)^{2} \right] - 2 \sum_{\mu} \sum_{\nu} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\nu)} M_{(\mu\nu)k} q^{k} = 0 \tag{5.6}$$

where

$$M_{(\mu\nu)k} = C_{ihk} \overset{*}{n}{}^{i}_{(\mu)} \overset{*}{n}{}^{h}_{(\nu)}. \tag{5.7}$$

Assuming

$$\sum_{\mu} \sum_{\nu} \overset{*}{K}_{(\mu)} \overset{*}{K}_{(\nu)} M_{(n\nu)k} q^{k} = 0 \tag{5.8}$$

and using the fact

$$\frac{\delta}{\delta s} \left[\sum_{\mu} \psi_{(\mu)} \left(\overset{*}{\Omega}_{(n)\alpha\beta} \frac{qu^{\alpha}}{ds} \frac{du^{\beta}}{ds} \right)^{2} \right] = \frac{d}{ds} \left[\sum_{\mu} \psi_{(\mu)} \left(\overset{*}{\Omega}_{(n)\alpha\beta} \frac{du^{\alpha}}{ds} \frac{du^{\beta}}{ds} \right)^{2} \right]$$

we get after integrating (5.6)

(5.9) $$\frac{1}{r_1^2} + \frac{1}{\varrho^2} = \frac{1}{a^2}$$

where $r_{(1)} = \frac{1}{k_{(1)}}$ and $\varrho = \left[\sum_{\mu} \psi_{(\mu)} \left(\overset{*}{\Omega}_{(\mu)\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right)^2 \right]^{-\frac{1}{2}}$ are the radii of geodesic (with respect to the subspace) and the secondary normal curvatures and $\frac{1}{a^2}$ is the constant of integration.

We have thereby established the followiug theorems:

Theorem (5.1). The sum of the squares of the geodesic and the secondary normal curvatures in the direction of a union hyper D-line is the same at all those points where the vector $M_{(\mu\nu)k}$ is orthogonal to the first curvature vector q^k:

Theorem (5.2). At all those points where the vector $M_{(\mu\nu)k}$ is orthogonal to q^k, the sum of squares of the geodesic and secondary normal curvatures in the direction of the union hyper D-line is the same relative to every congruence λ^i.

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ÖZET

Üç boyutlu bir Euklid uzayında bulunan bir yüzeye ait hiper DARBOUX çizgileri (hiper D-çizgileri) PRVANOVITCH tarafından tanımlanmış ve incelenmiştir [2]. SINGH, bir RIEMANN uzayının alt uzaylarındaki benzer çizgilerin bâzı özelliklerini incelemiştir [5]. Bu yazının gâyesi bir FINSLER uzayının bir alt uzayına ait hiper D-çizgilerinin diferansiyel denklemini elde etmek ve bundan, birleşim eğrileri olan hiper D-çizgilerinin bâzı özelliklerini bulmaktır.