SPECIAL CURVES OF A HYPERSURFACE OF A FINSLER SPACE

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The notion of *union curves* of a Riemannian hypersurface with respect to a vector-field λ, considered by SPRINGER [5], MISHRA [1], UPADHYAYA [2] and extended by SINHA [4] and SINGH [8] to FINSLER hypersurfaces, has been further specialised recently for Riemannian hypersurfaces by TSAGAS [6] who has considered K_λ - *curves*. An analogue of these is considered here on hypersurfaces of a FINSLER spaee and some properties of these curves are obtained.

1. Introduction. Union curves of a Riemannian hypersurface with respect to the vector-field λ were studied by SPRINGER [5], MISHRA [1], UPADHYAYA [7]. These curves (relative to a congruence λ) were studied by SINHA [4] and SINGH [8] on FINSLER hypersurfaces. Following the parallel definition, recently G. TSAGAS [6] studied the special curves (curves K_λ) of the Riemannian hypersurface. In the present paper, we wish to study the special curves K_λ in the hypersurface of a FINSLER space.

Consider a FINSLER space F_n of n-dimensions with coordinate system x^i $(i=1, 2,\dots,n)$ whose metric function $F(x, x')$ satisfies the conditions usually imposed upon a FINSLER metric [2]. The hypersurface F_{n-1} with coordinate system u^α $(\alpha = 1,2,\dots, n-1)$ is immersed in F_n and is given by the equations $x^i = x^i(u^\alpha)$ such that the rank of the matrix $\| B^i_\alpha \|$ is $n-1$. Let a curve $C: u^\alpha = u^\alpha(s)$, s being the arc length, be defined in F_{n-1} The components of its unit tangents $x'^i = dx^i/ds$ and $u'^\alpha = du^\alpha/ds$ with respect to F_n and F_{n-1} are related by

$$x'^i = B^i_\alpha u'^\alpha. \tag{1.1}$$

The metric tensors $g_{ij}(x, x')$ of F_n and $g_{\alpha\beta}(u, u')$ of F_{n-1} are connected by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') B^i_\alpha B^j_\beta. \tag{1.2}$$

In the geometry of FINSLER spaces, there exists two types of unit vectors normal to F_{n-1}. One is independent of the directional argument x'^i denoted by n^i and the other one is dependent upon x'^i and denoted by $n^{*i}(x, x')$. These vectors are defined by the equations [2]

$$n_j B^j_\alpha = g_{ij}(x, u)\, n^i B^j_\alpha = 0\ , \tag{1.3}$$

$$n^*{}_j B^j_\alpha = g_{ij}(x, x')\, n^{*i} B^j_\alpha = 0\ , \tag{1.4}$$

and are normalised by

$$g_{ij}(x, n)\, n^i n^j = n_j n^j = 1\ , \tag{1.5}$$

$$g_{ij}(x, n^*)\, n^{*i} n^{*j} = 1\ . \tag{1.6}$$

The vector n^{*i} also satisfies the equation

(1.7) $$g_{ij}(x, x') n^{*i} n^{*j} = \psi .$$

The covariant derivative $I^i_{\alpha\beta}$ of B^i_α is given by [2],

(1.8) $$I^i_{\alpha\beta} = \Omega_{\alpha\beta} n^i + B^i_\delta M^\delta \Omega_{\alpha\beta}$$

where

(1.9) $$M^\alpha = -n^i B^\alpha_i \quad , \quad B^\alpha_i B^i_\beta = \delta^\alpha_\beta .$$

$I^i_{\alpha\beta}$ is also given by

(1.10) $$I^i_{\alpha\beta} = \Omega^*_{\alpha\beta} n^{*i}$$

where $\Omega_{\alpha\beta}$ and $\Omega^*_{\alpha\beta}$ are the second fundamental tensor and secondary second fundamental tensor of the hypersurface.

Consider a congruence λ^i on F_{n-1} which is not necessarily normal to F_{n-1}. Since there are two types of normal vectors, we may consider two types of congruences. Let a congruence (independent of x'^i) at any point of F_{n-1} be expressed as

(1.11) $$\lambda^i(u) = t^\alpha(u) B^i_\alpha + \Gamma n^i(u) .$$

These vectors are normalised by the condition

(1.12) $$g_{ij}(x, \lambda) \lambda^i \lambda^j = 1 .$$

The congruence depending upon x'^i at any point of C on F_{n-1} is given by

(1.13) $$\lambda^{*i}(u, u') = t^{*\alpha}(u, u') B^i_\alpha + \Gamma^*(u, u') n^{*i}(u, u') .$$

Let these vectors be normalised by the equations

(1.14) $$g_{ij}(x, x') \lambda^{*i} \lambda^{*j} = 1 .$$

Using the equations (1.11) and (1.12), we have

(1.15) $$\begin{aligned} 1 &= g_{ij}(x, \lambda) \lambda^i \lambda^j \\ &= \bar{\gamma}_{\alpha\beta} t^\alpha t^\beta + 2\Gamma \bar{\gamma}_\alpha t^\alpha + \Gamma^2 \bar{\gamma} \end{aligned}$$

where we have written

(1.16) $$\bar{\gamma}_{\alpha\beta} = g_{ij}(x, \lambda) B^i_\alpha B^j_\alpha \quad , \quad \bar{\gamma}_\alpha = g_{ij}(x, \lambda) B^i_\alpha n^j \quad , \quad \bar{\gamma} = g_{ij}(x, \lambda) n^i n^j .$$

Again using the equations (1.2), (1.4), (1.7) and (1.13) in (1.14), we have

(1.17) $$1 = g_{ij}(x, x') \lambda^{*i} \lambda^{*j} = g_{\alpha\beta}(u, u') t^{*\alpha} t^{*\beta} + \Gamma^{*2} \psi .$$

In the following section, we shall consider the curves which will be called *special* or K_λ *- curves*.

2. Special Curves (K_λ - curves).

Definition (2.1). A curve on F_{n-1} is said to be a K_λ-curve if at any point of this curve, the vector λ^i (or λ^{*i}) tangent to a curve of the congruence lies in the surface determined by the geodesic curvature vectors of K_λ (or $K_\lambda{}^*$) - curve with respect to F_n and F_{n-1}.

Let C: $u^\alpha = u^\alpha(s)$ be any K_λ-curve on F_{n-1}, s being its arc length. By its definition, we have

$$\lambda^i = r\, p^i + s\, q^i \tag{2.1}$$

where λ^i is the vector at any point p of C and p^i and q^i are the geodesic curvature vectors of C at P with respect to F_{n-1} and F_n, as given by RUND [2],

$$\begin{cases} p^i = B^i_\alpha\, p^\alpha, \\ q^i = I^i_{\alpha\beta}\, u'^\alpha u'^\beta + B^i_\alpha\, p^\alpha. \end{cases} \tag{2.2}$$

and r and s are the parameters to be determined. It is also known that

$$p^\alpha = \delta u'^\alpha/\delta s = \frac{d^2 u^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma}\, u'^\beta u'^\gamma.$$

Using the equations (1.8), (1.11) and (2.2) in (2.1) we have

$$t^\alpha B^i_\alpha + \Gamma n^i = r\, B^i_\alpha\, p + s\,(B^i_\alpha\, p^\alpha + \Omega_{\alpha\beta}\, u'^\alpha u^\beta n^i + \Omega_{\beta\gamma}\, u'^\beta u'^\gamma M^\alpha B^i_\alpha).$$

Since n^i and B^i_α are the linearly independent vectors, we have

$$t^\alpha = (r+s)\, p^\alpha + s\, \Omega_{\beta\gamma}\, u'^\beta u'^\gamma M^\alpha \tag{2.4}$$

and

$$\Gamma = s\, \Omega_{\alpha\beta}\, u'^\alpha u'^\beta\,. \tag{2.5}$$

The equations (2.4) and (2.5) give

$$r = \left(t^\alpha - \frac{\Gamma p^\alpha}{K_n} - m^\alpha \Gamma\right)\Big/ p^\alpha \tag{2.6}$$

where

$$K_n = \Omega_{\alpha\beta}\, u'^\alpha u'^\beta\,.$$

Multiplying (2.4) by $\bar{\gamma}_{\alpha\gamma}\, t^\gamma$ and using (1.15), we get

$$(1 - 2\,\Gamma\, \bar{\gamma}_\alpha\, t^\alpha - \Gamma^2 \bar{\gamma} = (r+s)\, \bar{\gamma}_{\alpha\gamma}\, p^\alpha t^\gamma + s\, K_n\, \bar{\gamma}_{\alpha\gamma}\, M^\alpha t^\gamma. \tag{2.7}$$

Eliminating r and s from (2.4), (2.5) and (2.7) and simplifying, we have the required equation of the K_γ - curve given by

$$p^\alpha - (1 - 2\,\Gamma\, \bar{\gamma}_\alpha\, t^\alpha - F^2 \bar{\gamma})^{-1} \{(t^\alpha - \Gamma M^\alpha)\, \bar{\gamma}_{\beta\gamma}\, p^\beta t^\gamma + \Gamma\, \bar{\gamma}_{\gamma\beta}\, M^\beta t^\gamma\} = 0\,. \tag{2.8}$$

It may be noted that this equation reduces to a simpler form in a Riemannian hypersurface [6]. However, we may note the following :

Theorem (2.1). Let the congruence λ^i be not normal to F_{n-1}. The solutions of the system of $n-1$ differential equations (2.8) determines a K_λ - curve on F_{n-1} with respect to λ^i.

From the equation (2.8), we have :

Corollary (2.1). Each geodesic on the hypersurface F_{n-1} is a K_λ - curve.

If we denote the left hand side of (2.8) by a vector T^α, we may state that the K_λ - curve may be considered as a curve on F_{n-1} such that its vector T^α is null at any point of this curve.

The equation (2.8) is not much similar in the form to that of the equation in a Riemannian hypersurface. Now we shall find an equation of K^*_λ - curve in F_{n-1} which is very much similar to the corresponding equation on a Riemannian hypersurface.

By the definition of K^*_λ - curve, we may write

$$\lambda^{*i} = a\, p^i + b\, q^i \tag{2.9}$$

where λ^{*i} is a vector at P of C and p^i and q^i are given by (2.2) and a, b are the parameters.

The equation (2.9), by virtue of the equation (1.13) and (2.2), takes the form

$$t^{*\alpha} B^i_\alpha + \Gamma^* n^i = (a+b)\, B^i_\alpha\, p^\alpha + b\, I^i_{\alpha\beta}\, u'^\alpha u'^\beta\,. \tag{2.10}$$

Multiplication of (2.10) by $g_{ij}(x, x')\, B^j_\beta$ and summation with respect to i and use of relation (1.2) (1.4) and (1.10) yields the $n-1$ equations

$$g_{\alpha\beta}\, t^{*\alpha} = (a+b)\, g_{\alpha\beta}\, p^\alpha. \tag{2.11}$$

Multiplying this equation by $t^{*\beta}$ and summing on β, we have

$$g_{\alpha\beta}\, t^{*\alpha} t^{*\beta} = (a+b)\, g_{\alpha\beta}\, p^\alpha t^{*\beta}\,.$$

which by virtue of (1.17) gives

$$a + b = (1 - \Gamma^{*2}\psi)/g_{\alpha\beta}\, p^\alpha t^{*\beta}\,. \tag{2.12}$$

Multiplying (2.11) by $g^{\beta\gamma}$ and summing over β, we get

$$p^\alpha - t^{*\alpha}/(a+b) = 0$$

which by virtue of (2.12) takes the form

$$p^\alpha - t^{*\alpha} g_{\beta\gamma}\, p^\beta t^{*\gamma} (1 - \Gamma^{*2}\psi)^{-1} = 0\,. \tag{2.13}$$

In view of the equation (2.13), the theorems corresponding to the theorems (2.1) and the corollary corresponding to (2.1) are trivial.

Let us consider a curve $C: u^\alpha = u^\alpha(s)$ on F_{n-1}. At any point of C, consider the vector

$$T^{*\alpha} = p^\alpha - g_{\alpha\gamma}\, p^\beta t^{*\gamma} (1 - \Gamma^{*2}\psi)^{-1} t^{*\alpha}\,.$$

Definition (2.2). A curve on F_{n-1} is said to be a K^*_λ - curve if its vector $T^{*\alpha}$ is null at each point of this curve.

If K_{T*} be the magnitude of the vector $T^{*\alpha}$ given by

$$K^2_{T*} = g_{\alpha\beta}(u, u')\, T^{*\alpha}\, T^{*\beta}$$

then

$$K_{T*} = k_g \sin\theta$$

where $k^2_g = g_{\alpha\beta}(u, u')\, p^\alpha p^\beta$ and θ is the angle between the vectors p^α and $t^{*\alpha}$ such that

$$\cos\theta = \frac{g_{\alpha\beta}\, p^\alpha\, t^{*\beta}}{\sqrt{(g_{\alpha\beta}\, p^\alpha\, p^\beta)\,(g_{\gamma\delta}\, t^{*\gamma}\, t^{*\delta})}}\ .$$

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ÖZET

SPRINGER [5], MISHRA [1] ve UPADHYAYA [7] tarafından incelenen ve RIEMANN uzayında bulunan bir hiperyüzeyin bir λ vektör alanına bağlı *birleşim eğrileri* kavramı FINSLER uzayında bulunan hiperyüzeylere SINHA [4] ve SINGH [3] tarafından teşmil edilmiştir. Bu kavram yakın zamanlarda TSAGAS [6] tarafından daha da özelleştirilerek, RIEMANN uzayındaki hiperyüzeylerin K_λ - *eğrilerinin* incelenmesine yol açmıştır. Bu araştırmada, bir FINSLER uzayının hiperyüzeyleri için TSAGAS'ın K_λ - eğrilerinin benzerleri tanımlanmakta ve bunların bâzı özellikleri elde edilmektedir

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