ON THE GENERALISED LIE DIFFERENTIATION ARISING FROM SU'S INFINITESIMAL TRANSFORMATION

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The results of *generalised* LIE *differentiation* in FINSLER space, defined in [6] by using SU's infinitesimal transformation [10], are applied to BERWALD and RUND's connection parameters, the generalised CHRISTOFFEL symbols and projective connection parameters. Certain commutation rules concerning the generalised LIE operator and various kinds of operators which arise in the theory of FINSLER spaces are also obtained.

The theory of LIE differentiation of Riemannian geometry was extended to FINSLER geometry by DAVIES [2] and LAPTEW [8]. These authors considered the infinitesimal transformation $\bar{x}^i = x^i + \varepsilon v^i(x^j)$ 1) where the vector-field v^i, as in Riemannian geometry, is a function of the coordinates x^i and ε is an infinitesimal constant. Later SU [10] generalised the above transformation taking v^i as a function of line-elements $(x^i, \dot{x}^i)$ 2) and extended the results of DAVIES concerning geodesic deviation. Corresponding to SU's infinitesimal transformation 3) the differentiation analogous to LIE differentiation (so called generalised LIE differentiation or briefly GL-differentiation) has been defined by the present author and MISHRA [6]. The GL-derivatives of vectors, tensors and CARTAN's connection parameters have been derived in that paper. The further aspects of GL-differentiation are studied in the present paper. The first section is introductory and includes some of the known results for their applications in our analysis. The second, third and fourth sections deal with the GL-differentiation of BERWALD's and RUND's connection parameters, the symbols analogous to those of CHRISTOFFEL (so called generalised CHRISTOFFEL symbols or GC-symbols) and the projective connection parameters. In the fifth section it is seen that the operator $\dot{\partial}_i = \partial/\partial\dot{x}^i$ which is commutative over the LIE differential operator (called briefly LIE operator) with respect to a RI-transformation is no more commutative over the GL-operator with respect to a GL-transformation. Lastly, in the sixth section, the commutation rules of the GL-operator over various kinds of covariant differentiation operators have been derived. Unless stated otherwise the notation used in this paper is based on [8] and [9].

1. Preliminaries. Let F_n be an n-dimensional FINSLER space with the metric function $F(x, \dot{x})$ of class at least C^5. The entities defined by $g_{ij}(x, \dot{x}) \overset{\text{def}}{=} (1/2)\, \dot{\partial}_i\, \dot{\partial}_j\, F^2$ and $g^{ik}\, g_{kj} = \delta^i_j$ form the components of a tensor called the metric tensor of F_n and are symmetric and positively

1) Latin indices run from 1 to n throughout the paper.

2) Henceforth the line-element $(x^i, \dot{x}^i)$ will be briefly denoted by $(x, \dot{x})$.

3) An infinitesimal transformation will be called RI-(restricted infinitesimal) transformation if v^i is a function of the x's and GI-(generalised infinitesimal) transformation if v^i is a function of the $(x, \dot{x})$.

homogeneous of degree zero in the $\dot{x}^i$'s. The connection parameters Γ^{*i}_{jk} (of CARTAN) and G^i_{jk} (of BERWALD) are expressed in terms of the tensors $C^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} g^{ih} C_{jhk} = \frac{1}{2} g^{ih} \dot{\partial}_j g_{hk}$ and the GC-symbols γ^i_{jk} are connected by

$$\Gamma^{*i}_{jk} \dot{x}^k = G^i_{jk} \dot{x}^k = G^i_j \overset{\text{def}}{=} \dot{\partial}_j G^i \overset{\text{def}}{=} \dot{\partial}_j \left(\frac{1}{2} \gamma^i_{lm} \dot{x}^l \dot{x}^m \right) . \tag{1.1}$$

Replacing the element of support $\dot{x}^i$ by an arbitrary vector-field $\xi^i(x^j)$ RUND introduced a set of connection parameters $P^{*i}_{jk}(x, \xi)$ which ultimately coincide with those of CARTAN.

Two kinds of CARTAN's covariant derivatives of a tensor T are given by [1])

$$\nabla_k \mathrm{T} = \partial_k \mathrm{T} - (\dot{\partial}_j \mathrm{T}) G^j_k + \sum_\alpha T^{\cdots r \cdots}_{\cdots\cdots} \Gamma^{*i_\alpha}_{rk} - \sum_\beta T^{\cdots\cdots}_{\cdots r \cdots} \Gamma^{*r}_{j_\beta k} , \tag{1.2}$$

and

$$\dot{\nabla}_k \mathrm{T} = \dot{\partial}_k \mathrm{T} + \sum_\alpha T^{\cdots r \cdots}_{\cdots\cdots} C^{i_\alpha}_{rk} - \sum_\beta T^{\cdots\cdots}_{\cdots r \cdots} C^r_{j_\beta k}. \tag{1.3}$$

The difference $(\nabla_j \nabla_k - \nabla_k \nabla_j)$ T gives rise to the curvature tensor

$$K^i_{jkh} = 2 \left\{ \partial_{[j} \Gamma^{*i}_{k]h} + \left(\dot{\partial}_r \Gamma^{*i}_{h[j} \right) G^r_{k]} + \Gamma^{*i}_{r[j} \Gamma^{*r}_{k]h} \right\}^{1)}, \tag{1.4}$$

where the square brackets denote the skew-symmetric part with respect to the indices enclosed within them. [2]) BERWALD's covariant derivative $\mathfrak{B}_k$ T and the curvature tensor $H^i_{jkh}(x, \dot{x})$ may be described by replacing Γ^{*i}_{jk} by G^i_{jk} in (1.2) and (1.4). Indeed, BERWALD constructed the curvature tensor H^i_{jkh} in the following way :

$$H^i_{jkh} \overset{def}{=} \dot{\partial}_h H^i_{jk} , \quad H^i_{jk} \overset{def}{=} \frac{2}{3} \dot{\partial}_{[j} H^i_{j]} , \tag{1.5}$$

where $H^i_k(x, \dot{x})$ is the deviation tensor. These tensors satisfy

$$H^i_{jkh} \dot{x}^h = H^i_{jk} , \quad H^i_{jk} \dot{x}^k = H^i_j . \tag{1.6}$$

Similarly the partial δ-derivative and the corresponding curvature tensor $\widetilde{K}^i_{jkh}(x, \dot{x})$ of RUND [9] (henceforth called after him) may be obtained from (1.2) and (1.4) by replacing the functions $-G^j_k$ by the derivatives $\partial_k \xi^j$.

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1) The covariant differential operators $|\, k$, (k) and $;k$ given by CARTAN, BERWALD and RUND [9] are more elegantly denoted by ∇_k, $\mathfrak{B}_k$ and R_k respectively. Also the derivative $\mathrm{T}\,|_k$ ([9], equation (4.1.20)) is suitably modified in (1.3).

2) The present K^i_{jkh} is same as K^i_{hkj} of [9], equation (4.1.7) and R^i_{jkh} of [11], p. 187.

The GL-derivatives of a tensor $\mathbf{T}$ and the connection parameters Γ^{*i}_{jk} with respect to the GI-transformation

(1.7) $$\bar{x}^i = x^i + \varepsilon v^i(x, \dot{x}), \quad \dot{\bar{x}}^i = \dot{x}^i + \varepsilon(\dot{x}^j \partial_j v^i + \ddot{x}^j \dot{\partial}_j v^i)$$

are given by [6]

(1.8a) $$\mathfrak{L}\,\mathbf{T} = v^k \partial_k \mathbf{T} - \sum_\alpha T^{\cdots k \cdots}_{\cdots\cdots\cdots} \partial_k v^{i_\alpha} + \sum_\beta T^{\cdots\cdots\cdots}_{\cdots k \cdots} \partial_{j_\beta} v^k + (\dot{\partial}_k \mathbf{T})(\dot{x}^h \partial_h v^k + \ddot{x}^h v^k{}_h),$$

(1.8b) $$\mathfrak{L}\mathbf{T} = v^k \nabla_k \mathbf{T} - \sum_\alpha T^{\cdots k \cdots}_{\cdots\cdots\cdots} (\nabla_k v^{i_\alpha} + v^{i_\alpha}_h G^h_k) + \sum_\beta T^{\cdots\cdots\cdots}_{\cdots k \cdots} (\nabla_{j_\beta} v^k + v^k_h G^h_{j_\beta}) + (\dot{\partial}_k \mathbf{T})(\dot{x}^h \nabla_h v^k + w^k),$$

(1.9a) $$\mathfrak{L}\Gamma^{*i}_{jk} = \partial_j \partial_k v^i + v^h \partial_h \Gamma^{*i}_{jk} - \Gamma^{*h}_{jk} \partial_h v^i + 2\,\Gamma^{*i}_{h(j}\partial_{k)} v^h + (\dot{\partial}_h \Gamma^{*i}_{jk})(\dot{x}^r \partial_r v^h + \ddot{x}^r v^h_r),$$

(1.9b) $$\mathfrak{L}\Gamma^{*i}_{jk} = \nabla_j \nabla_k v^i + v^h K^i_{hjk} + (\dot{\partial}_h \Gamma^{*i}_{jk})(\dot{x}^r \nabla_r v^h + w^h) + (\partial_k v^i_h + \Gamma^{*i}_{kr} v^r_h) G^h_j + \nabla_j (v^i_h G^h_k),\ ^{1)}$$

where $v^i_j \overset{\text{def}}{=} \dot{\partial}_j v^i$, $w^i(x, \dot{x}) \overset{\text{def}}{=} v^i_j(\ddot{x}^j + 2G^j)$ and the round brackets denote the symmetric part.

2. ***GL*-derivatives of G^i_{jk} and P^{*i}_{jk}.** As the connection parameters of Berwald possess the same transformation law as those of Cartan therefore for $\mathfrak{L}\, G^i_{jk}$ we may transcribe almost word for word the method used to derive the formula (1.9b). Thus we may obtain

(2.1) $$\mathfrak{L}G^i_{jk} = \mathfrak{B}_j \mathfrak{B}_k v^i + v^h H^i_{hjk} + G^i_{hjk}(\dot{x}^r \mathfrak{B}_r v^h + w^h) + (\partial_k v^i_h + G^i_{kr} v^r_h) G^h_j + \mathfrak{B}_j (v^i_h G^h_k),$$

where $G^i_{hjk} \overset{\text{def}}{=} \dot{\partial}_h G^i_{jk}$ is a tensor-field symmetric in all the lower indices.

From (1.1) we have

(2.2) $$\mathfrak{L}G^i_j = (\mathfrak{L}G^i_{jk})\,\dot{x}^k + G^i_{jk} f \dot{x}^k$$

and

(2.3) $$\mathfrak{L}\,G^i = (\mathfrak{L}\,G^i_j)\,\dot{x}^j + G^i_j\, \mathfrak{L}\dot{x}^j .$$

1) For convenience the GL-operator $\underset{L}{D}$ of [6] has been replaced by $\mathfrak{L}$.

Substituting from (2.1) and

$$\mathfrak{L}\dot{x}^k = \ddot{x}^h v_h^k \tag{2.4}$$

in (2.2) and using (1.1), (1.6), $\mathfrak{B}_j \dot{x}^k = 0$ and the homogeneity properties of G_{jk}^i we derive

$$\begin{aligned} \mathfrak{L}G_j^i = & \mathfrak{B}_j \mathfrak{B}_k v^i \dot{x}^k + v^h H_{hj}^i + (\dot{x}^k \partial_k v_h^i + G_r^i v_h^r) G_j^h \\ & + 2\mathfrak{B}_j (v_h^i G^h) + G_{jk}^i v_h^k \ddot{x}^h \end{aligned} \tag{2.5}$$

Also , from (2.3), (2.4) and (2.5), we derive

$$\begin{aligned} \mathfrak{L}G^i = & \frac{1}{2}(\mathfrak{B}_j \mathfrak{B}_k v^i \dot{x}^j \dot{x}^k + v^h H_h^i + G_k^i v_h^k \ddot{x}^h) \\ & + (\dot{x}^k \partial_k v_h^i + G_r^i v_h^r) G^h + \mathfrak{B}_j (v_h^i G^h) \dot{x}^j . \end{aligned} \tag{2.6}$$

For, RUND's connection parameters P_{jk}^{*i} being identical with those of CARTAN $\mathfrak{L} P_{jk}^{*i}$ may be evaluated from the equation (1.8a). However, if we require to express $\mathfrak{L} P_{jk}^{*i}$ in aid of RUND's covariant derivative and the corresponding curvature tensor $\widetilde{K}_{jkh}^i$ we note that the RUND's covariant differentiation differs with that of CARTAN in the respect that the former involves the derivatives $\partial_j \xi^i$, which are purely functions of x^i's only, instead of the functions $-G_j^i(x, \dot{x})$ of the latter. The same distinction also appears in the corresponding curvature tensors $\widetilde{K}_{jkh}^i$ and K_{jkh}^i. The derivatives $\mathfrak{R}_j \mathfrak{R}_k v^i$ and $\nabla_j \nabla_k v^i$ also differ in the same way except for a term resulting from the differentiation of G_k^h with respect to $\dot{x}^r$ which, in fact, vanishes in the case of $\mathfrak{R}_k v^i$. Therefore to express $\mathfrak{L} P_{jk}^{*i}$ in the required form we have to add one term, namely $v^h (\partial_h \xi^m) \dot{\partial}_m P_{jk}^{*i}$ and subtract the same in (1.9a). Thus, we derive

$$\begin{aligned} \mathfrak{L}P_{jk}^{*i} = & \mathfrak{R}_j \mathfrak{R}_k v^i + v^h \widetilde{K}_{hjk}^i + (\dot{\partial}_h P_{jk}^{*i})(\dot{x}^r \partial_r v^h + \ddot{x}^r v_r^h - v^r \partial_r \xi^h) \\ & - (\partial_k v_h^i + P_{kr}^{*i} v_h^r) \partial_j \xi^h - \mathfrak{R}_j (v_h^i \partial_k \xi^h). \end{aligned} \tag{2.7}$$

Eliminating $\partial_r v^h$ by the definition of $\mathfrak{R}_r v^h$, introducing one extra term $2 v_r^h G^r$ and applying (1.1) and the definition of w^i, the coefficient of $\dot{\partial}_h P_{jk}^{*i}$ in the above equation reduces to

$$\dot{x}^r \mathfrak{R}_r v^h + w^h - v_r^h (\partial_l \xi^r + P_{lm}^{*r} \dot{x}^m) \dot{x}^l - v^l (\partial_l \xi^h + P_{lm}^{*h} \dot{x}^m).$$

Now taking the vector - field ξ^i as the element of support $\dot{x}^i$ and applying the definition of $R_l \xi^i$ it further reduces to

$$\dot{x}^r \mathfrak{R}^l v^h + w^h - (v_h^r \dot{x}^l + v^l \delta_h^r) \mathfrak{R}_l \xi^r.$$

Consequently the equation (2.7) may be rewritten as

$$\mathfrak{L}P^{*i}_{\ k} = \mathfrak{R}_j\, \mathfrak{R}_k\, v^i + v^h\, \widetilde{K}^i_{hjk} - (\partial_h\, v^i_h + P^{*i}_{kr}\, v^r_h)\, \partial_j\, \xi^h$$

(2.8)

$$+ (\dot\partial_h\, P^{*i}_{jk})\,\{\dot x^r\, \mathfrak{R}_r\, v^h + w^h - (v^h_r\, \dot x^l + v^l\, \delta^h_r)\, R_l\, \xi^r\} - R_j\,(v^i_h\, \partial_k\, \xi^h).$$

When the vector-field ξ^i is regarded as stationary[1]) the covariant differentiation process of RUND becomes identical with that of CARTAN and the corresponding curvature tensors coincide. Consequently the equation (2.8) coincides with (1.9b).

3. *GL*-differentiation of *GC*-symbols. It has been seen in [5] that the formula for the LIE derivative of Γ^{*i}_{jk} with respect to a *RI*-transformation can not be applied to evaluate the LIE derivative of γ^i_{jk} because they do not satisfy the same transformation law as Γ^{*i}_{jk}. Therefore to evaluate the LIE derivative of γ^i_{jk} we had to start from the definition. Similarly to evaluate the *GL*-derivative of *GC*-symbols with respect to a *GI*-transformation we have to revert to the definition.

Firstly, we notice that the variation in γ^i_{jk} under the *GI*-transformation (1.7) is given by

$$\gamma^i_{jk}(\bar x, \dot{\bar x}) = \gamma^i_{jk}(x, \dot x) + \varepsilon\,\{v^h\, \partial_h\, \gamma^i_{jk} + (\dot\partial_h\, \gamma^i_{jk})\,(\dot x^m\, \partial_m\, v^h + \dot x^m\, v^h_m)\}.$$

Eliminating $\partial_m\, v^h$ by means of

(3.1)
$$v^h_{,m} = \partial_m\, v^h - v^h_l\, \gamma^l_{mr}\, \dot x^r + v^l\, \gamma^h_{lm}\,, \text{ [2]})$$

and using (1.1) the above equation reduces to

(3.2)
$$\gamma^i_{jk}(\bar x, \dot{\bar x}) = \gamma^i_{jk}(x, \dot x) + \varepsilon\,\{v^h\, \partial_h\, \gamma^i_{jk} + (\dot\partial_h\, \gamma^i_{jk})\,(\dot x^m\, v^h_{,m} + w^h - v^l\, \gamma^h_{lm}\, \dot x^m)\}.$$

Secondly, writing the transformation law for γ^i_{jk} ([5], equation (2.1)) the variation in γ^i_{jk} when the *GI*-transformation (1.7) is regarded a coordinate transformation is obtained by

$$'\gamma^i_{jk}(\bar x, \dot{\bar x}) = \gamma^i_{jk}(x, \dot x) - \varepsilon\,[\partial_j\, \partial_k\, v^i + 2\, \gamma^i_{h(j}\, \partial_{k)}\, v^h - \gamma^h_{jk}\, \partial_h\, v^i$$

(3.3)

$$+ \{2\, C^i_{h(j}\, \partial_{k)}\, \partial_l\, v^h - g^{im}\, C_{jkh}\, \partial_l\, \partial_m\, v^h\}\, \dot x^l].$$

Thus, from (3.2) and (3.3), we get

$$\mathfrak{L}\gamma^i_{jk} = \partial_j\, \partial_k\, v_k\, v^i + 2\, \gamma^i_{h(j}\, \partial_{k)}\, v^h - \gamma^h_{jk}\, \partial_h\, v^i + v^h\, \partial_k\, \gamma^i_{jk}$$

(3.4)

$$+ (\dot\partial_h\, \gamma^i_{jk})\,(\dot x^m\, v^h_{,m} + w^h - v^l\, \gamma^h_{lm}\, \dot x^m)$$

$$+ \{2\, C^i_{h(j}\, \partial_{k)}\, \partial_l\, v^h - g^{im}\, C_{jkh}\, \partial_l\, \partial_m\, v^h\}\, \dot x^l.$$

Applying the definition of $v^i_{,jk}$ and the curvature tensor type quantities Z^i_{jkh} (arising from the covariant differentiation for γ^i_{jk}) the above formula may be reduced to

[1]) cf. [9, 96.]

[2]) The covariant differentiation with respect to γ^i_{jk} has been defined in [5, equation (2.3)].

$$\mathfrak{L}\gamma^i_{jk} = v^i_{,jk} + v^h Z^i_{jkh} + (\partial_h \gamma^i_{jk})(\dot{x}^m v^h_{,m} + w^h)$$

(3.5)
$$+ \{(\partial_j v^i_h + \gamma^i_{hr} v^r_h)\gamma^h_{km} + (v^i_h \gamma^h_{jm})_{,k}$$

$$+ 2 C^i_{h(j} \partial_{k)} \partial_m v^h - g^{ir} C_{jkh} \partial_r \partial_m v^h\} \dot{x}^m.$$

4. ***GL*-derivative of projective connection parameters.** Recently the present author and MEHER [8] derived the formula for the LIE derivative of the projective connection parameters

(4.1)
$$\pi^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} G^i_{jk} - \frac{1}{n+1}\{2\delta^i_{(j} G^r_{k)r} + x^i G^r_{jkr}\}$$

for a *RI*-transformation. The variation in π^i_{jk} under the *GI*-transformation (1.7) is given by

(4.2)
$$\pi^i_{jk}(\bar{x}, \dot{\bar{x}}) = \pi^i_{jk}(x, \dot{x}) + \varepsilon\{v^h \pi_h \pi^i_{jk} + \pi^i_{hjk}(\dot{x}^r \pi_r v^h + \ddot{x}^r v^h_r)\}.$$

Also the transformation law for π^i_{jk} is

$$'\pi^i_{jk}(\bar{x}, \dot{\bar{x}}) = \{\pi^a_{bc}(\bar{\partial}_j x^b)(\bar{\partial}_k x^c) + \bar{\partial}_j \bar{\partial}_k x^a\}(\pi_a \bar{x}^i)$$

$$+ \frac{2}{n+1}(\pi_b \pi_a \bar{x}^r)\delta^i_{(j}(\bar{\partial}_{k)} x^b)(\bar{\partial}_r x^c),$$

which, for the *GI*-transformation, simplifies to

$$'\pi^i_{jk}(\bar{x}, \dot{\bar{x}}) = \pi^i_{jk}(\bar{x}, \dot{\bar{x}}) + \frac{2\varepsilon}{n+1}\delta^i_{(j} \pi_{k)} \pi_h v^h$$

(4.3)
$$- \varepsilon\{\pi_j \pi_k v^i + 2\pi^i_{h(j} \pi_{k)} v^h - \pi^h_{jk} \pi_h v^i\}.$$

From (4.2) and (4.3) we thus have

$$\mathfrak{L}\pi^i_{jk} = \partial_j \partial_k v^i + v^h \partial_h \pi^i_{jk} + 2\pi^i_{h(j} \partial_{k)} v^h - \pi^h_{jk} \partial_h v^i$$

(4.4)
$$- \frac{2}{n+1}\delta^i_{(j} \partial_{k)} \partial_h v^h + \pi^i_{hjk}(\dot{x}^r \partial_r v^h + \ddot{x}^r v^h_r).$$

Applying the definition of projective covariant differentiation [1]) and the definition of projective type quantities Q^i_{jkh} the above equation may also be written as

$$\mathfrak{S}\pi^i_{jk} = P_j P_k v^i + v^h Q^i_{hjk} + \pi^i_{hjk}(x P_r v^h + \omega^h)$$

(4.5)
$$+ (\partial_j v^i_h + \pi^i_{rj} v^r_h)\pi^h_k + P_k(v^i_h \pi^h_j) - \frac{2}{n+1}\delta^i_{(j} \partial_{k)} \partial_h v^h$$

where

(4.6)
$$\omega^h(x, x) \overset{\text{def}}{=} v^h_r(\ddot{x}^r + 2\pi^r)$$

together with

1) cf. [4] and [8].

(4.7) $$2\pi^r = \pi^r_j \dot{x}^j = \pi^r_{jk} \dot{x}^j \dot{x}^k.$$

It may be noted that the entities ω^h are different than ν^h of the first section.

5. The operators $\mathfrak{L}$, $\dot{\partial}_k$ and ∂_k. The operator $\dot{\partial}_k$ has been seen ([11], p. 189) commutative over the Lie operator with respect to an *RI*-transformation. But in the following we observe that it is not commutative over the *GL*-operator with respect to the *GI*-transformation (1.7).

We enunciate the

Theorem 5.1. *For the quantities* [1] $\Omega^{\cdots}_{\cdots}(x, \dot{x})$ *the operators* $\dot{\partial}_h$ *and* $\mathfrak{L}$ *commute according to*

(5.1) $$(\dot{\partial}\,\mathfrak{L} - \partial\dot{\mathfrak{L}}_h)\,\Omega^{\cdots}_{\cdots} \dot{f}_h\,\Omega^{\cdots}_{\cdots},$$

where $\dot{\mathfrak{L}}_h$ *stands for the GL-derivative with respect to a GI-transformation generated by the tensor field* $\dot{\partial}_h v^i$.

Proof. First we consider a vector-field $X^i(x, \dot{x})$ whose *GL*-derivative may be obtained from (1.8a). Differentiating $\mathfrak{L}\, x^i$ with respect to $\dot{x}^h$ and noting that $\ddot{x}^k$ are independent of $\dot{x}^h$ we have

(5.2)
$$\begin{aligned}\dot{\partial}_h\, \mathfrak{L}X^i = (\dot{\partial}_h v^j)(\partial_j X^i) + v^i \partial_j \dot{\partial}_h X^i - (\dot{\partial}_h x^j)(\partial_j v^i)\\ - X^j \partial_j \dot{\partial}_h v^i + (\dot{\partial}_j \dot{\partial}_h X^i)(\dot{x}^k \partial_k v^j + \ddot{x}^k \dot{\partial}_k v^j)\\ + (\dot{\partial}_j X^i)(\partial_h v^j + \dot{x}^k \partial_k \dot{\partial}_h v^j + \ddot{x}^k \dot{\partial}_k \dot{\partial}_h v^j).\end{aligned}$$

Further, $\dot{\partial}_h X^i$ being a tensor-field its *GL*-derivative may be found from (1.8a).

(5.3)
$$\begin{aligned}\mathfrak{L}\dot{\partial}_h X^i = v^j \partial_j \dot{\partial}_h X^i - (\partial_h X^j)\partial_j v^i + (\dot{\partial}_j X^i)\,\partial_h v^j\\ + (\dot{\partial}_j \dot{\partial}_h X^i)(\dot{x}^k \partial_h v^j + \ddot{x}^k \dot{\partial}_k v^j).\end{aligned}$$

From (5.2) and (5.3) we obtain

$$\begin{aligned}(\dot{\partial}_h\, \mathfrak{L} - \mathfrak{L}\, \dot{\partial}_h)\, x^i = (\dot{\partial}_h v^j)\,\partial_j X^i - X^j \partial_j (\dot{\partial}_h v^i)\\ + (\dot{\partial}_j X^i)\{\dot{x}^k \partial_k (\dot{\partial}_h v^j) + \ddot{x}^k \dot{\partial}_k (\dot{\partial}_h v^j)\}.\end{aligned}$$

Comparison of the second member of this identity with the expression for $\mathfrak{L}X^j$ shows that the former might have been derived from the latter replacing v^i by $\dot{\partial}_h v^i$. Thus, the formula (5.1) holds for a vector-field.

Similarly it may be verified for a scalar and a tensor also.

To establish (5.1) for the connection parameters Γ^{*i}_{jk} or P^{*i}_{jk} we differentiate (1.9a) with respect to $\dot{x}^h$ and then evaluate the *GL*-derivative of $\dot{\partial}_h\, \Gamma^{*i}_{jk}$ from (1.8a) as these derivatives constitute a tensor-field. Thus, the expression for $(\dot{\partial}_h\, \mathfrak{L} - \mathfrak{L}\, \dot{\partial}_h)\,\Gamma^{*i}_{jk}$, after cancelling some common terms in the operations indicated, reduces to one derived from the *GL*-derivative of by writing $\dot{\partial}_h v^i$ for v^i. Further, the *GL*-derivative of G^i_{jk} is analogous to that of Γ^{*i}_{jk} we can verify (5.1) for G^i_{jk} in a similar way.

[1] By quantities here we mean scalars, vectors, tensors, connection parameters of Rund, Cartan and Berwald, *GC*-symbols and the projective connection parameters .

Next, to verify (5.1) for the *GC*-symbols a little complication arises. Since $\dot{\partial}_h \gamma^i_{jk}$ do not, in general, form the components of a tensor, so $\mathfrak{L}\, \dot{\partial}_h \gamma^i_{jk}$ can not be directly calculated from (1.8a). Starting from the definition we may find

$$\begin{aligned} \mathfrak{L}\, \dot{\partial}_h \gamma^i_{jk} &= v^l\, \partial_l\, \dot{\partial}_h\, \gamma^i_{jk} + (\dot{\partial}_l\, \dot{\partial}_h\, \gamma^i_{jk})\,(\dot{x}^m\, \partial_m\, v^l + \ddot{x}^m\, \dot{\partial}_m\, v^l) \\ &+ 2\,\{\dot{\partial}_h\, \gamma^i_{l(j}\}\, \partial_{k)}\, v_l + (\dot{\partial}_h\, \gamma^l_{jk})\, \partial_h\, v^l - \dot{\partial}_h\, \gamma^l_{jk})\, \partial_l\, v^i + 2\, C^i_{l(j}\, \partial_{k)}\, \partial_h\, v^l \\ &- g^{il}\, C_{jkm}\, \partial_l\, \partial_h\, v^m + [2\,\{\dot{\partial}_h\, C^i_{l(j}\}\, \partial_{k)}\, \partial_m\, v^l - (\dot{\partial}_h\, g^{ir}\, C_{jkl})\, \partial_r\, \partial_m\, v^l]\, \dot{x}^m. \end{aligned} \tag{5.4}$$

Then differentiating (3.4) with respect to $\dot{x}^h$ and forming the difference $(\dot{\partial}_h\, \mathfrak{L} - \mathfrak{L}\, \dot{\partial}_h)\, \gamma^i_{jk}$ we obtain, after cancelling some common terms, the expression for it in the form $\mathfrak{L}_h\, \gamma^i_{jk}$.

Finally, it is easy to verify (5.1) for the projective connection parameters as their derivatives with respect to $\dot{x}^h$ do constitute a tensor-field symmetric in all the lower indices. Thus, differentiating (4.4) with respect to $\dot{x}^h$ and subtracting the expression for $\mathfrak{L}\, \dot{\partial}_h\, \partial^i_{jk}$ from the resulting equation we get

$$(\dot{\partial}_h\, \mathfrak{L} - \mathfrak{L}\, \dot{\partial}_h)\, \partial^i_{jk} = \dot{f}_h\, \partial^i_{jk}\,. \tag{5.5}$$

This completes the proof of the above theorem.

Note 5.1. For an *RI*-transformation $\dot{\mathfrak{L}}_h\, \Omega^{\cdots}_{\cdots}$ vanishes and therefore the operator $\dot{\partial}_h$ is commutative over the LIE derivative.

Theorem 5.2. *For any vector-field $X^i(x\,, \dot{x})$ the operators $\mathfrak{L}$ and $\dot{\partial}_k$ commute according to*

$$(\mathfrak{L}\dot{\partial}_k - \dot{\partial}_k\, \mathfrak{L})\, x^i = (\dot{\partial}_j\, X^i)\, (\dot{x}^h\, \partial_k\, \partial_h\, v^j + \ddot{x}^h\, \partial_k\, v^j_h). \tag{5.6}$$

Proof: As the derivatives $\dot{\partial}_k\, X^i$ do not, in general, possess the tensorial characteristics their *GL*-derivatives can not be calculated from (1.8a). Starting from the definition we derive

$$\begin{aligned} \mathfrak{L}\, \dot{\partial}_k\, X^i &= v^i\, \partial_j\, \dot{\partial}_k X^i - (\dot{\partial}_k\, X^j)\, \partial_j\, v^i + (\dot{\partial}_j\, X^i)\, \partial_k\, v^j \\ &- x^j\, \partial_j\, \partial_k\, v^i + (\dot{\partial}_j\, \dot{\partial}_k\, X^i)\, (\dot{x}^h\, \partial_h\, v^j + \ddot{x}^h\, \dot{\partial}_h\, v^j). \end{aligned} \tag{5.7}$$

Also, differentiating $\mathfrak{L}X^i$ with respect to $\dot{x}^k$ and subtracting the resulting equation from (5.7) we get (5.6).

6. Commutation rules. In general the covariant differentiation is not commutative over the LIE differentiation even with respect to an *RI*-transformation. In the following theorems we derive rules for the *GI*-operator $\mathfrak{L}$ to commute over various processes of covariant differentiation.

Theorem 6.1. *For any vector-field $X^i\,(x\,, \dot{x})$ the opertors $\mathfrak{L}$ and Δ_k commute according to*

$$(\mathfrak{L}\nabla_k - \nabla_k\, \mathfrak{L})\, X^i = X^j\, \mathfrak{L}\, \Gamma^{*i}_{jk} - (\dot{\partial}\, X^i)\, (\mathfrak{L}\, G^j_k + \partial_k \mathfrak{L}\, \dot{x}^j) - G^j_k \dot{\mathfrak{L}}_j\, X^i\,. \tag{6.1}$$

Proof. $\nabla_k\, X^i$ being a tensor, its *GL*-derivative may be found from (1.8b).

$$\mathfrak{L}\nabla_k X^i = v^i \nabla_j \nabla_k X^i - (\nabla_k X^j)(\nabla_j v^i + v^i_h G^h_j)$$

(6.2)

$$+ (\nabla_j X^i)(\nabla_k v^j + v^j_h G^h_k) + (\dot\partial_j \nabla_k X^i)(\dot x^h \nabla_h v^j + w^j).$$

Now applying the covariant differential operator ∇_k over the GL-derivative of X^i we get

$$\nabla_k \mathfrak{L} X^i = v^j \nabla_k \nabla_j X^i + (\nabla_k v^j)\nabla_j X^i - (\nabla_k X^j)(\nabla_j v^i + v^i_h G^h_j)$$

(6.3)

$$- X^j \{\nabla_k \nabla_j v^i + \nabla_k (v^i_h G^h_j)\} + (\nabla_k \dot\partial_j X^i)(\dot x^h \nabla_h v^j + w^j$$

$$+ (\dot\partial_j X^i)(\dot x^h \nabla_k \nabla_h v^j + \nabla_k w^j),$$

where we have used the fact that $\dot x^h$ remains invariant under ∇_k. Subtracting (6.3) from (6.2) and using the commutation formulae

$$2\nabla_{[j}\nabla_{k]} X^i = K^i_{jkh} X^h - (\dot\partial_l X^i) K^l_{jkh} \dot x^h$$

and

$$(\dot\partial_j \nabla_k - \nabla_k \dot\partial_j) X^i = X^h \dot\partial_j \Gamma^{*i}_{kh},$$

we get

$$(\mathfrak{L}\Delta_k - \Delta_k \mathfrak{L}) X^i = X^j \{\Delta_k \nabla_j v^i + v^h K^i_{hkj} + (\partial_h \Gamma^{*i}_{kj})(\dot x^r \nabla_r v^h + w^h) + \nabla_k (v^i_h G^h_i)\}$$

(6.4)

$$+ (\nabla_j X^i) v^j_h G^h_k - (\dot\partial_j X^i)(\nabla_k \nabla_h v^j \dot x^h + v^l K^j_{lkh} \dot x^h + \nabla_k w^j).$$

In view of (2.4) we may write

(6.5) $$\nabla_k w^j = \nabla_k \mathfrak{L}\dot x^j + 2\nabla_k (v^j_h G^h).$$

Thus the relation (6.4), in consequence of (1.9p), (2.5) and (6.5), reduces to

$$(f\nabla_k - \nabla_k f) x^i = x^i f \Gamma^{*l}_{jk} - (\partial_j x^i)(f G^j_k + \partial_k f \dot x^j)$$

$$+ \left\{(\partial_j x^i) v^j_h - x^j \partial_j v^j_h + (\partial_j x^i)(\dot x^r \partial_r v^j_h + \ddot x^r \partial_r v^j_h)\right\} G^k_h .$$

Now applying the definition of the operator $\dot{\mathfrak{L}}_h$ this relation becomes identical with (6.2).

The GL-operator $\mathfrak{L}$ commutes over BERWALD's covariant differential operator $\mathfrak{B}_k$ in a way analogous to (6.1) :

(6.6) $$(\mathfrak{L}\mathfrak{B}_k - \mathfrak{B}_k \mathfrak{L}) x^i = x^j \mathfrak{L} G^i_{jk} - (\dot\partial_j x^i)(\mathfrak{L} G^j_k + \partial_k \mathfrak{L}\dot x^j) + G^j_k \mathfrak{L}_j x^i .$$

Theorem 6.2. *The operators* $\mathfrak{L}$ *and* $\dot\nabla_k$ *commute according to*

(6.7) $$(\mathfrak{L}\dot\nabla_k - \dot\nabla_k \mathfrak{L})\; x^i = x^j \mathfrak{L}\, G^i_{jk} \dot{\mathfrak{L}}_k - x^i$$

Proof. The derivative $\dot\nabla_k x^i$ being a tensor, its GL-derivative may be obtained from (1.8a). Thus, applying (1.3) we evaluate $\mathfrak{L}\dot\nabla_k x^i$:

$$\mathfrak{L}\dot{V}_k\, x^i = v^j \left\{\partial_j\, \dot{\partial}_k\, x^i + (\partial_j\, x^h)\, C^i_{kh} + x^h\, \partial_j\, C^i_{kh}\right\}$$

(6.8)

$$+ \left\{ \dot{\partial}_j\, \dot{\partial}_k\, x^i + (\partial_j\, x^h\, C^i_{kh} + x^h\, \dot{\partial}_j\, C^i_{kh}\right\} \dot{x}^r\, \partial_r\, v_j + \ddot{x}^r\, v^j_r\,)$$

Next, writing the expression for $\mathfrak{L}\, x^i$ by (1.8a) and applying the operator $\dot{V}_k$ over it we also eavluate $\Delta_k\, \mathfrak{L}\, x^i$:

$$\dot{V}_k\, \mathfrak{L}x^i = v^j_k\, \partial_j\, x^i + v_j\, \dot{\partial}_k\, x^i - (\dot{\partial}_k\, x^j)\, \dot{\partial}_j\, v^i - x^j\, \partial_j\, \partial_k\, v^i$$

(6.9) $$+ (\dot{\partial}_j\, \dot{\partial}_k\, x^i)\, (x^h\, \partial_h\, v^j + \ddot{x}^h\, v^i_h\,) + (\dot{\partial}^j\, x^i)\, (\partial_k\, v^j + \dot{x}^h\, \dot{\partial}_k\, \partial_h\, v^j + \ddot{x}^h\, \dot{\partial}_h v^j_k$$

$$+ \left\{v^h\, \partial_h\, x^j - x^h\, \partial_h\, v^j + (\dot{\partial}_h\, x^j)\, (\dot{x}^r\, \partial_r\, v^h + \ddot{x}^r\, v^h_r\,)\right\} C^i_{jk}\, .$$

Thus the formula (6.7) follows from (6.8) and (6.9).

Theorem 6.3. *The processes of GL-differentiation and the projective covariant differentiation commute according to*

(6.10) $$(\mathfrak{L}\mathfrak{P}_k - \mathfrak{P}_k)\, X^i = X^j\, \mathfrak{L}\pi^i_{jk} - (\partial_j\, X^i)\, (\mathfrak{L}\, \partial^{\,j}_{\,k} + \dot{\partial}_k\, \mathfrak{L}\, x^j) + \pi^j_k\, \dot{\mathfrak{L}}_j\, X^j.$$

Proof. Since the projective covariant derivative $\mathfrak{P}_k X^i x$ does not, in general, possess the tensor character its GL-derivative can not be directly obtained from (1.8a). Therefore, to evaluate $\mathfrak{L}\, \mathfrak{P}_k\, X^i$ we use the relation ([4], equation (2.3)).

$$\mathfrak{P}_k\, X^i = B_k\, X^i + \frac{1}{n+1}\, [\dot{\partial}_j\, X^i)\, (\, \delta^j_k\, G^r_r + \dot{x}^j\, G^r_{rk}\,) - X^j \left\{ 2\, \delta^i_{j)}\, G^r_{k)r} + \dot{x}^i\, G^r_{rjk}\right\}]$$

Thus we have

$$\mathfrak{L}\mathfrak{P}_k\, X^i = \mathfrak{L}B_k\, X^i + \frac{1}{n+1}\, [(\mathfrak{L}\dot{\partial}_j\, X^i)\, (\delta^j_k\, G^r_r + \dot{x}^j\, G^r_{kr})$$

(6.11) $$+ (\partial_j\, X^i) \left\{ \delta^j_k\, \mathfrak{L}G^r_r + \mathfrak{L}(\dot{x}^j\, G^r_{kr}) \right\} \left\{- (\mathfrak{L}\, X^j) \right\} 2\, \delta^i_j\, G^r_{k)r} + \dot{x}^j\, G^r_{jkr}\Big\}$$

$$- X^j \left\{ 2\, \delta^i_{(j}\, \mathfrak{L}G^r_{k)r} + \mathfrak{L}(x^i\, G^r_{jkr}\,) \right\}].$$

Also the projective covariant derivative of $\mathfrak{L}X^i$ given by

(6.12) $$\mathfrak{P}_k\, \mathfrak{L}X^i = B_k\, \mathfrak{L}X^i + \frac{1}{n+1}\, [(\partial_j\, \mathfrak{L}X^i\,)\, (\delta\, G^r_r + \dot{x}^j\, G^r_{rk}\,)$$

$$- (\mathfrak{L}X^j) \left\{ 2\, \delta^i_{(j}\, G^r_{k)r} + \dot{x}^i\, G \right\}].$$

Subtracting (6.12) from (6.11) and using (4.1), (5.1), (6.6) we derive (6.10) after some simplification.

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ÖZET

RIEMANN geometrisindeki LIE türev alma teorisi DAVIES [2] ve LAPTEW [3] tarafından FINSLER uzaylarına teşmil edilmiştir. Bu iki yazar

$$\bar{x}^i = x^i + \varepsilon\, v^i\,(x^j)$$

şeklinde bir infinitezimal dönüşümü hareket noktası olarak kullanmışlardır. Ancak, RIEMANN geometrisinde olduğu gibi, v^i vektör alanını x^j nokta koordinatlarının bir fonksiyonu ve ε nu sonsuz küçük bir sabit olarak almışlardır. Daha sonra SU [10] bu dönüşümü, vektör alanını FINSLER uzayının bir $(x^i, \dot{x}^j)$ lineer elemanının fonksiyonu olarak düşünmek suretiyle teşmil etmiş ve DAVIES'in bâzı sonuçlarını genelleştirmiştir. SU'nun dönüşümünden hareket edilerek *genelleştirilmiş* LIE *türev alma* işlemi MISHRA ve yazar tarafından [6] tanımlanmıştır. Bu araştırmada genelleştirilmiş LIE türev alma işleminin bâzı uygulamaları üzerinde durulmaktadır. Bu işlemin BERWALD ve RUND'un bağımlılık parametreleri, genelleştirilmiş CHRISTOFFEL sembolleri ve projektif bağımlılık parametreleri ile ilgisi incelenmekte ve işleme tekabül eden operatör ile FINSLER uzayları teorisinde karşılaşılan diğer bâzı operatörler için komütasyon bağıntıları elde edilmektedir.