GENERALISED GAUSS-CODAZZI EQUATIONS FOR BERWALD'S CURVATURE TENSORS IN SUBSPACES OF A FINSLER SPACE

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The object of this paper is to give a further extension of the GAUSS-CODAZZI to some subspaces of a FINSLER space.

Introduction. SINHA [1][1]) has obtained the generalised GAUSS and CODAZZI equations for BERWALD's curvature tensors in a hypersurface of a FINSLER space. In the present paper, the same equations have been derived in the case of a congruence of curves associated to a subspace of a FINSLER space and some particular cases have been discussed.

1. Notations and basic concepts. In order to explain the notations and to clarify the concepts used below some basic formulae of the theory of FINSLER spaces and their subspaces are briefly presented here. Consider a FINSLER space F_n of n dimensions referred to a local coordinate system x^i $(i = 1, 2, \ldots, n)$, whose metric function $F(x^i, \dot{x}^i)$ satisfies the conditions usually imposed upon it [2, cb. I].

The metric tensor of F_n is defined by $g_{ij}(x, \dot{x}) = \frac{1}{2}\, \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$[2]), and since this is positively homogeneous of degree one in $\dot{x}^k$, the tensor $C_{ijk}(x, \dot{x}) = \frac{1}{2}\, \dot{\partial}_k g_{ij}(x, \dot{x})$ satisfies the identities,

$$C_{ijk}(x, \dot{x})\, \dot{x}^i = C_{ijk}(x, \dot{x})\, \dot{x}^j = C_{ijk}(x, \dot{x})\, \dot{x}^k = 0\,.$$

Let the parametric equations of the subspace F_m of a FINSLER space F_n be represented by $x^i = x^i(u^\alpha)$, $(\alpha = 1, 2, \ldots, m)$, where u^α are the parameters of the subspace. It will be assumed throughout that the function x^i are at least of class C^3, and that the matrix of the projection parameters $\| B^i_\alpha \|$ has rank m.

We shall use the following notations:

$$B^i_\alpha = \partial_\alpha x^i\,, \qquad B^i_{\alpha\beta} = \partial_\alpha \partial_\beta x^i\,, \qquad B^{ij\ldots k}_{\alpha\beta\ldots\gamma} = B^i_\alpha B^j_\beta \cdots B^k_\gamma\,.$$

A subspace vector $\dot{u}^\alpha$ possesses components $\dot{x}^i$ with respect to the coordinate system F_n which are given by $\dot{x}^i = B^i_\alpha \dot{u}^\alpha$. The induced fundamental tensor of F_m is given by

1) The numbers in the square brackets refer to the references given in the end.

2) $\partial_i = \partial/\partial x^i$, $\quad \dot{\partial}_i = \partial/\partial \dot{x}^i$.

(1.1) $$g_{\alpha\beta}(u^{\gamma}, \dot{u}^{\gamma}) = g_{ij}(x^k, \dot{x}^k)\, B^{ij}_{\alpha\beta}\,.$$

In terms of the tensor $g^{\alpha\beta}$, which is the inverse of (1.1), one may introduce the quantities

(1.2) $$B^{\beta}_{i} = g^{\alpha\beta} g_{ij} B^{j}_{\alpha}$$

and these in turn are the inverse of B^i_α : $B^i_\alpha B^\beta_i = \delta^\beta_\alpha$. Since the rank of the $\| B^i_\alpha \|$ is assumed to be m, it follows that there exists a field of $(n - m)$ linearly independent vectors $N^i_{(\nu)}$ normal to F_m and they may be chosen in a multiply infinite number of ways, given by the relations

(1.3) $$N_{(\mu)i}\, B^i_\alpha = g_{ij}\, N^j_{(\mu)}\, B^i_\alpha = 0, \quad (\nu, \sigma, \mu = m+1, \ldots, n).$$

The set of vectors are normalised by means of the relations

$$N^i_{(\nu)} = g^{ij}(x, \dot{x})\; N_{(\nu)j}\,,$$

$$g_{ij}(x, \dot{x})\, N^i_{(\mu)} N^j_{(\nu)} = \cos(N_{(\mu)}, N_{(\nu)}) = a_{(\mu\nu)}\,.$$

The following tensors vanish identically in any locally Euclidean (or Riemannian) theory of subspaces :

$$M_{(\nu)ij}(x, \dot{x}) = C_{ijk}(x, \dot{x})\, N^k_{(\nu)}\,,$$

$$M_{(\nu)i} = \sum_{\sigma} M_{(\sigma\nu)ij}\, N^j_{(\sigma)}\,, \quad M^i_{(\nu)j} = g^{ih}\, M_{(\nu)jh}\,.$$

The connection coefficients of L. BERWALD [2, ch. III] are denoted by G^i_{jk} and are used to define a covariant derivative. For instance, the covariant derivative of a covariant tensor $T_{ij}(x, \dot{x})$ of degree two is given by

(1.4) $$T_{ij/k} = \partial_k T_{ij} - \dot{\partial}_r T_{ij}\, \dot{\partial}_k G^r - T_{rj}\, G^r_{ik} - T_{ir}\, G^r_{jk}\,.$$

Since the connection coefficient G^i_{hk} is homogeneous of degree zero in $\dot{x}^l$ we have

$$\dot{\partial}_l\, G^i_{hj}\, \dot{x}^h = 0\,.$$

The BERWALD and CARTAN connections are related by the relations

(1.5) $$G^k_{ij} = \Gamma^{*k}_{ij} + C^k_{ij|r}\, \dot{x}^r\,.$$

2. Induced and intrinsic derivatives on the subspace. As remarked by RUND [2], since the induced and intrinsic BERWALD connections are identical, the induced parameters $G^\varepsilon_{\alpha\beta}$ satisfy the relation,

(2.1) $$G^\varepsilon_{\alpha\beta} = \widehat{\Gamma}^{*\varepsilon}_{\alpha\beta} + C^\varepsilon_{\alpha\beta|\sigma}\, \dot{u}^\sigma\,.$$

Let

(2.2) $$\widehat{\Gamma}^{\varepsilon}_{\alpha\beta} = \Lambda^\varepsilon_{\alpha\beta} + \Gamma^{*\varepsilon}_{\alpha\beta}$$

where we have used the notations as in SINHA [1]. The induced symmetric tensors and connection parameters of F_m are given respectively by

(2.3) $$A^{\alpha}_{\beta\gamma} = A^{i}_{jk}\, B^{\alpha}_{i}\, B^{jk}_{\beta\gamma}$$

and

(2.4) $$\Gamma^{*\varepsilon}_{\alpha\beta} = B^{\varepsilon}_{i}\,(B^{i}_{\alpha\beta} + \Gamma^{*i}_{hk}\, B^{hk}_{\alpha\beta}).$$

With the help of these quantities we can define two types of induced mixed derivatives denoted by $T^{i}_{\alpha,\beta}$ and $T^{i}_{\alpha\|\beta}$ and defined by

(2.5) $$T^{i}_{\alpha,\beta} = F\dot{\partial}_{\beta}\, T^{i}_{\alpha} + T^{r}_{\alpha}\, A^{i}_{rj}\, B^{j}_{\beta} - T^{i}_{\gamma}\, A^{\gamma}_{\alpha\beta}\,,$$

(2.6) $$T^{i}_{\alpha\|\beta} = \partial_{\beta}\, T^{i}_{\alpha} - \dot{\partial}_{\gamma}\, T^{i}_{\alpha}\, \Gamma^{*\gamma}_{\beta\rho}\, \dot{u}^{\rho} + T^{r}_{\alpha}\, L^{*i}_{jr}\, B^{j}_{\beta} - T^{i}_{\rho}\, \Gamma^{*\rho}_{\alpha\beta}\,.$$

Substituting (2.2) and (2.3) in (2.1) we have

(2.7) $$G^{\varepsilon}_{\alpha\beta} = B^{\varepsilon}_{i}\,(B^{i}_{\alpha\beta} + \Gamma^{*i}_{hk}\, B^{hk}_{\alpha\beta}) + A^{\varepsilon}_{\alpha\beta} + C^{\varepsilon}_{\alpha\beta|\rho}\, \dot{u}^{\rho}\,.$$

With the help of (2.7) and (1.5) we have,

$$G^{\varepsilon}_{\alpha\beta} = B^{\varepsilon}_{i}\,[B^{i}_{\alpha\beta} + B^{hk}_{\alpha\beta}\,(G^{i}_{hk} - C^{i}_{hk|r}\, \dot{x}^{r})] + A^{\varepsilon}_{\alpha\beta} + C^{\varepsilon}_{\alpha\beta|\rho}\, \dot{u}^{\rho}.$$

These quantities allow us to define the following mixed tensor :

(2.8) $$V^{i}_{\alpha\beta} = B^{i}_{\alpha\beta} - B^{i}_{\varepsilon}\, G^{\varepsilon}_{\alpha\beta} + G^{i}_{hk}\, B^{hk}_{\alpha\beta}\,.$$

Simplifying (2.8) with the help of (1.5) and (2.1) we get,

$$V^{i}_{\alpha\beta} = J^{i}_{\alpha\beta} - B^{i}_{\varepsilon}\, C^{\varepsilon}_{\alpha\beta|\gamma}\, \dot{u}^{\gamma} + B^{hk}_{\alpha\beta}\, C^{i}_{hk|r}\, \dot{x}^{r}$$

where

$$J^{i}_{\alpha\beta} = B^{i}_{\alpha\beta} - B^{i}_{\varepsilon}\, \widehat{\Gamma}^{\varepsilon}_{\alpha\beta} + \Gamma^{*i}_{hk}\, B^{hk}_{\alpha\beta}\,.$$

The induced derivative $N^{i}_{(\nu)\|\gamma}$ of the type (2.6) can be easily obtained as

(2.9) $$N^{i}_{(\nu)\|\beta} = -\,\Omega_{(\nu)\alpha\beta}\, g^{\alpha\varepsilon}\, B^{i}_{\varepsilon} + E^{i}_{(\nu)l}\, J^{l}_{\rho\beta}\, \dot{u}^{\rho}$$

where

$$E^{i}_{(\nu)l} = \sum_{\sigma} N^{i}_{(\sigma)}\, M_{(\sigma\nu)l} - 2M^{i}_{(\nu)l}\,.$$

We shall in particular require the covariant derivative of the unit normal $N^{i}_{(\nu)}$ of F_m defined by

$$N^{i}_{(\nu)|\gamma} = \partial_{\gamma}\, N^{i}_{(\nu)} - \dot{\partial}_{\gamma}\, N^{i}_{(\nu)}\, \dot{\partial}_{\gamma}\, G^{\lambda} + N^{h}_{(\nu)}\, G^{i}_{hk}\, B^{k}_{\gamma}$$

which by virtue of (1.5) becomes

(2.10) $$N^i_{(\nu)|\gamma} = N^i_{(\nu)||\gamma} + N^h_{(\nu)} C^i_{hk|r} \dot{x}^r B^k_\gamma .$$

On substituting the value of $N^i_{(\nu)||\gamma}$ from (2.9) in (2.10) we obtain,

(2.11) $$N^i_{(\nu)|\gamma} = - \Omega_{(\nu)\alpha\gamma} g^{\alpha\varepsilon} B^i_\varepsilon + E^i_{(\nu)l} J^l_{\rho\gamma} \dot{u}^\rho + N^h_{(\nu)} C^i_{hk|r} \dot{x}^r B^k_\gamma .$$

3. The Generalised Gauss and Codazzi Equations. Consider a set of $(n-m)$ congruences of curves such that one curve of each of them passes through every point of F_m. We consider the contravariant component of a unit vector in the direction of a curve of a congruence of cures as expressed linearly in terms of B^i_α and set of the normals to F_m :

(3.1) $$V^i_{(\mu)} = t^\alpha_{(\mu)} B^i_\alpha + \sum_\nu d_{(\nu,\mu)} N^i_{(\nu)} .$$

In order to derive the GAUSS and CODAZZI equations we must evaluate the «mixed» derivatives of $V^i_{(\mu)}$ with respect to u^β, noting that the mixed derivatives $V^i_{\alpha\beta} = B^i_{\alpha/\beta}$. In view of (1.4) the value of $\lambda^i_{(\mu)/[\beta\gamma]}$ is obtained as

(3.2) $$\lambda^i_{(\mu)/[\beta\gamma]} = \tfrac{1}{2} H^i_{hkj} \lambda^h_{(\mu)} B^{kj}_{\beta\gamma} - \tfrac{1}{2} H^\varepsilon_{\delta\beta\gamma} \dot{\partial}_\varepsilon \lambda^i_{(\mu)} \dot{u}^\delta + \dot{\partial}_l G^i_{hk} V^l_{\varepsilon[\gamma} B^k_{\beta]} \lambda^h_{(\mu)} \dot{u}^\varepsilon$$

where H^i_{hkj}, $H^\varepsilon_{\alpha\beta\gamma}$ represent the components of the curvature tensors of F_n and F_m respectively, the first of these being defined by,

$$H^i_{h\,kj} = 2\,(\partial_{[j} G^i_{k]h} + G^l_{h[k} G^i_{j]l} + G^i_{rh[j} G^r_{k]}) .$$

The mixed covariant derivative of $V^i_{(\mu)}$ is given by

(3.3) $$\lambda^i_{(\mu)/\beta} = t^\alpha_{(\mu)} V^i_{\alpha\beta} + t^\alpha_{(\mu)/\beta} B^i_\alpha + \sum_\nu d_{(\nu,\mu)/\beta} N^i_{(\nu)} + \sum_\nu d_{(\nu,\mu)} N^i_{(\nu/\beta)} .$$

Again taking the mixed covariant derivative of (3.3) with respect to u^γ and considering the skew symmetric part in β and γ in the resulting equation, we get,

$$\lambda^i_{(\mu)/[\beta\gamma]} = t^\alpha_{(\mu)} V^i_{\alpha[\beta/\gamma]} + B^i_\alpha t^\alpha_{(\mu)/[\beta\gamma]} + \sum_\nu d_{(\nu,\mu)} N^i_{(\nu)/[\beta\gamma]} + \sum_\nu N^i_{(\nu)} d_{(\nu,\mu)/[\beta\gamma]} .$$

Taking the mixed covariant derivative of (2.11), simplifying and considering the skew symmetric part in β and γ we have,

$$N^{i}_{(\mu)/[\beta\gamma]} = -g^{\alpha\varepsilon}\, B^{i}_{\varepsilon}\, \Omega_{(\mu)\alpha[\beta/\gamma]} - \Omega_{(\mu)\alpha[\beta}\, V^{i}_{\gamma]\varepsilon}\, g^{\alpha\varepsilon} + E^{i}_{(\mu)l/[\gamma}\, V^{l}_{\beta]\rho}\, \dot{u}^{\rho} + E^{i}_{(\mu)l}\, V^{l}_{\varepsilon[\beta/\gamma]}\, \dot{u}^{\varepsilon} + E^{i}_{(\mu)}\, V^{l}_{\varepsilon[\beta}\, \dot{u}^{\varepsilon}_{/\gamma]} + (N^{h}_{(\mu)}\, C^{i}_{hk|r}\, \dot{x}^{r})_{/[\beta}\, B^{k}_{\gamma]} + N^{h}_{(\mu)}\, C^{i}_{hk|r}\, \dot{x}^{r}\, V^{k}_{[\beta\gamma]}\,. \tag{3.4}$$

Similarly we have the following relations:

$$2\, t^{\alpha}_{(\mu)/[\beta\gamma]} = t^{\delta}_{(\mu)}\, H^{\alpha}_{\delta\beta\gamma} - \dot{\partial}_{\varepsilon}\, t^{\alpha}_{(\mu)}\, H^{\varepsilon}_{\rho\beta\gamma}\, \dot{u}^{\rho} \tag{3.5}$$

and

$$2\, V^{i}_{\alpha[\beta/\gamma]} = B^{i}_{\varepsilon}\, H^{\varepsilon}_{\alpha\beta\gamma} + B^{hkj}_{\alpha\beta\gamma}\, H_{hkj} + \dot{\partial}_{l}\, G^{i}_{hk}\, B^{h}_{\alpha}\, (B^{k}_{\beta}\, V^{l}_{\varepsilon\gamma} - B^{k}_{\gamma}\, V^{l}_{\varepsilon\beta})\, \dot{u}^{\varepsilon}. \tag{3.6}$$

Simplifying (3.2) with the help of (3.4), (3.5) and (3.6) we obtain

$$\begin{aligned} H^{i}_{h\,kj}\, (\lambda^{h}_{(\mu)}\, B^{kj}_{\beta\gamma} - t^{\alpha}_{(\mu)}\, B^{hkj}_{\alpha\beta\gamma}) &= H^{\varepsilon}_{\delta\beta\gamma}\, (\dot{\partial}_{\varepsilon}\, \lambda^{i}_{(\mu)} - B^{i}_{\alpha}\, \dot{\partial}_{\varepsilon}\, t^{\alpha}_{(\mu)})\, \dot{u}^{\delta} \\ &- 2\, \dot{\partial}_{l}\, G^{i}_{hk}\, V^{l}_{\varepsilon[\gamma}\, B^{k}_{\beta]}\, \dot{u}^{\varepsilon}\, (\lambda^{h}_{(\mu)} - t^{\alpha}_{(\mu)}\, B^{h}_{\alpha}) + 2 \sum_{\nu} d_{(\nu,\mu)} \\ &\{ -g^{\alpha\varepsilon}\, B^{i}_{\varepsilon}\, \Omega_{(\nu)\alpha[\beta/\gamma]} - \Omega_{(\nu)\alpha[\beta}\, V^{i}_{\gamma]\varepsilon}\, g^{\alpha\varepsilon} \\ &+ E^{i}_{(\nu)l/[\gamma}\, V^{l}_{\beta]\rho}\, \dot{u}^{\rho} + E^{i}_{(\nu)l}\, V^{l}_{\varepsilon[\beta/\gamma]}\, \dot{u}^{\varepsilon} + E^{i}_{(\nu)}\, V^{l}_{\varepsilon[\beta}\, \dot{u}^{\varepsilon}_{/\gamma]} \} \\ &+ (N^{h}_{(\nu)}\, C^{i}_{hk|r}\, \dot{x}^{r})_{/[\beta}\, B^{k}_{\gamma]} + N^{h}_{(\nu)}\, C^{i}_{hk|r}\, \dot{x}^{r}\, V^{k}_{[\beta\gamma]} \\ &+ 2 \sum_{\nu} N^{i}_{(\nu)}\, d_{(\nu,\mu)/[\beta\gamma]}\,. \end{aligned} \tag{3.7}$$

Multiplying (3.7) by $g_{im}\, B^{m}_{\lambda}$ and using (1.1), (1,2) and (1.3) we get,

$$\begin{aligned} H_{hmkj}\, (\lambda^{h}_{(\mu)}\, B^{m\,kj}_{\lambda\,\beta\gamma} - t^{\alpha}_{(\mu)}\, B^{m\,h\,k\,j}_{\lambda\,\alpha\,\beta\,\gamma}) &= H^{\varepsilon}_{\delta\beta\gamma}\, (\dot{\partial}_{\varepsilon}\, \lambda^{i}_{(\mu)}\, g_{im}\, B^{m}_{\lambda} - \dot{\partial}_{\varepsilon}\, t^{\alpha}_{(\mu)}\, g_{\alpha\lambda})\, \dot{u}^{\delta} \\ &- 2\, \dot{\partial}_{l}\, G^{i}_{hk}\, V^{l}_{\varepsilon[\gamma}\, B^{k}_{\beta]}\, \dot{u}^{\varepsilon}\, (\lambda^{h}_{(\mu)} - t^{\alpha}_{(\mu)}\, B^{h}_{\alpha})\, g_{im}\, B^{m}_{\lambda} \\ &- 2 \sum_{\nu} d_{(\nu,\mu)}\, \{ g^{\alpha\varepsilon}\, B^{i}_{\varepsilon}\, \Omega_{(\nu)\alpha[\beta/\gamma]}\, g_{im}\, B^{m}_{\lambda} \\ &+ \Omega_{(\nu)\alpha[\beta}\, V^{i}_{\gamma]\varepsilon}\, g^{\alpha\varepsilon}\, B_{im}\, B^{m}_{\lambda} - E^{i}_{(\nu)l/[\gamma}\, V^{l}_{\beta]\rho}\, \dot{u}^{\rho}\, g_{im}\, B^{m}_{\lambda} \\ &- E^{i}_{(\nu)l}\, V^{l}_{\varepsilon[\beta/\gamma]}\, \dot{u}^{\varepsilon} + E^{i}_{(\nu)}\, V^{l}_{\varepsilon[\beta}\, \dot{u}^{\varepsilon}_{/\gamma} \} + (N^{h}_{(\nu)}\, C^{i}_{hk|r}\, \dot{x}^{r})_{/[\beta}\, B^{k}_{\gamma}). \end{aligned} \tag{3.8}$$

Again multiplying (3.7) by $N_{(\sigma)i}$ and simplifying as above, we have,

$$(3.9)\qquad H^i_{h\,kj}(\lambda^h_{(\mu)}\,B^{kj}_{\beta\gamma}-t^\alpha_{(\mu)}\,B^{h\,kj}_{\alpha\beta\,\gamma})\,N_{(\sigma)i}$$

$$=H^\varepsilon_{\delta\beta\gamma}\,\dot\partial_\varepsilon\,\lambda^i_{(\mu)}\,\dot u^\delta-2\,\dot\partial_l\,G^i_{hk}\,V^l_{\varepsilon[\gamma}\,B^k_{\beta]}\,\dot u^\varepsilon\,N_{(\sigma)i}(\lambda^h_{(\mu)}-t^\alpha_{(\mu)}\,B^h_\alpha)$$

$$-2\sum_\nu d_{(\nu,\mu)}\,[\Omega_{(\nu)\alpha[\beta}\,V^i_{\gamma]\varepsilon}\,g^{\alpha\varepsilon}\,N_{(\sigma)i}-E^i_{(\nu)l|\gamma}\,V^l_{\beta]\rho}\,N_{(\sigma)i}\,\dot u^\rho$$

$$-E^i_{(\nu)l}\,V^l_{\varepsilon[\beta/\gamma]}\,\dot u^\varepsilon\,N_{(\sigma)i}-E^i_{(\nu)}\,V^l_{\varepsilon[\beta}\,\dot u^\varepsilon_{/\gamma]}]$$

$$+(N^h_{(\nu)}\,C^i_{hk|r}\,\dot x^r)_{/[\beta}\,B^k_{\gamma]}\,N_{(\sigma)i}+N^h_{(\nu)}\,N^l_{(\sigma)}\,C^l_{hk|r}\,\dot x^r\,V^k_{|\beta\gamma]}$$

$$+2\sum_\nu d_{(\nu,\mu)/[\beta\gamma]}\,a_{(\nu,\sigma)}\,.$$

Equations (3.8) and (3.9) which are based on a vector $\lambda^i_{(\mu)}$ of a most general nature can be regarded as generalisation of the GAUSS-CODAZZI equations in a subspace F_m imbedded in a FINSLER space F_n .

4. Particular cases. The congruence of curves can be considered in the following three ways :

(*i*) The vector $\lambda^i_{(\mu)}$ in F_n lies in a space spanned by the normals, that is,

$$\lambda^i_{(\mu)}=\sum_\nu d_{(\mu,\nu)}\,N^i_{(\nu)}\,.$$

(*ii*) The vector $\lambda^i_{(\mu)}$ lies in the space spanned by B^i_α , that is,

$$\lambda^i_{(\mu)}=t^\alpha_{(\mu)}\,B^i_\alpha\,.$$

(*iii*) The vector $\lambda^i_{(\mu)}$ is tangential to the curves,

$$\dot x^i=\dot u^\alpha\,B^i_\alpha\,.$$

These equations can easily be obtained for the hypersurfaces of a FINSLER space as well.

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ÖZET

Bu araştırmanın gâyesi, GAUSS-CODAZZI denklemlerini FINSLER uzaylarının bâzı altuzaylarına bir genellemektir.