SUBHARMONIC FUNCTION OF INFINITE ORDER IN THE HALF-PLANE

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Abstract. In this study, it is proved that if a proper subharmonic function of infinite order have full measure at the finite system of rays in the upper half-plane then its lower order also equals to infinity.

1. Introduction

Let v be a subharmonic function in the complex plane $\mathbb{C}$ and $M(v,r) = \max\limits_{0\le\theta\le 2\pi} v(re^{i\theta})$. The following values

$$\beta[\gamma] = \limsup_{r\to\infty} \frac{\ln M(v,r)}{\ln r}, \quad \alpha[\gamma] = \liminf_{r\to\infty} \frac{\ln M(v,r)}{\ln r}$$

are called *an order and lower order* of the function v, respectively. *An order and lower order of an entire function* f are called an order and lower order of subharmonic function $\ln|f|$, respectively.

In [**1**] the entire functions whose zeros lie on the finite system of rays were considered. In particular, it was proved that if f is an entire function of infinite order with positive zeros then its lower order also equals to infinity. This result is easily generalized to the subharmonic functions in the complex plane: if the measure of Riesz of the subharmonic function in the entire complex plane v of infinite order is located on a positive half-axis then its lower order also equals to infinity. We prove the similar result for functions which are subharmonic in the half-plane.

2. Classes of functions in the upper half-plane

Let $\mathbb{C}_+ = \{z : \Im z > 0\}$ be the upper half-plane of the complex variable z. We denote the open disc of radius r with center at a by $C(a,r)$ and the intersection of a set Ω with the half-plane $\mathbb{C}_+$ by Ω_+: $\Omega_+ = \Omega \cap \mathbb{C}_+$. $\overline{G}$ means closure of a set G. If $0 < r_1 < r_2$ then $D_+(r_1, r_2) = \overline{C_+(0,r_2)\backslash C_+(0,r_1)}$ means close half-ring.

Let SK be the class of subharmonic functions in $\mathbb{C}_+$ processing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. Functions $v(z)$ in SK have the following properties [**2**]:

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a) $v(z)$ has non-tangential limits $v(t)$ almost everywhere on the real axis and $v(t) \in L^1_{loc}(-\infty, \infty)$;

b) There exists a measure ν on the real axis such that

$$\lim_{y\to+0} \int\limits_a^b v(t+iy)\,dt = \nu([a,b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\})\,.$$

The measure ν is called *the boundary measure* of v;

c) $d\nu(t) = v(t)\,dt + d\sigma(t)$, where σ is a singular measure with respect to Lebesgue measure.

For a function $v \in SK$, following [**2**], we define *the full measure* λ as

$$\lambda(K) = 2\pi \int\limits_{\mathbb{C}_+\cap K} \Im\zeta\, d\mu(\zeta) - \nu(K)\,,$$

where μ is the Riesz measure of v.

A subharmonic function v in $\mathbb{C}_+$ is called *proper subharmonic* if $\limsup_{z\to t} v(z) \le 0$ for all real numbers $t \in \mathbb{R}$. Denote the class of proper subharmonic functions by JS. The full measure of the function $v \in JS$ is a positive measure, which stands for the term "proper subharmonic function".

The class of delta-subharmonic functions $J\delta$ is defined as a difference $J\delta = JS - JS$.

The following statements are true [**2**]:

Statement 1. $JS \subset SK$.

Statement 2. $J\delta = SK - SK$.

From Statement 2, it follows that $SK \subset J\delta$. So further we may consider the subharmonic functions of the class JS because the functions of the class SK are represented as difference of two proper subharmonic functions.

For function $v \in J\delta$ the following representation in a disc $z \in C_+(0,R)$ holds:

$$v(z) = -\frac{1}{2\pi}\iint_{\overline{C_+(0,R)}} \frac{G(z,\zeta)}{\Im\zeta}\,d\lambda(\zeta) + \frac{R}{2\pi}\int_0^\pi \frac{\partial G(z,Re^{i\varphi})}{\partial n} v(Re^{i\varphi})\,d\varphi\,, \tag{1}$$

where $G(z,\zeta)$ is the Green function of the half-disc, $\dfrac{\partial G}{\partial n}$ means a derivative in the inward normal direction, and the kernel of double integral is extended by continuity to the real axis for $|t| \le R$.

For the measure λ denote $\lambda(t) = \lambda(\overline{C(0,t)})$. Let $v \in J\delta$, $v = v_+ - v_-$ and λ is the full measure of v. The Jordan decomposition of measure λ is $\lambda = \lambda_+ - \lambda_-$. Let us introduce the following characteristics of the function v:

$$m(r,v) := \frac{1}{r}\int_0^\pi v_+(re^{i\varphi})\sin\varphi\,d\varphi, \quad N(r,v,r_0) := \int_{r_0}^r \frac{\lambda_-(t)}{t^3}\,dt\,,$$
$$T(r,v,r_0) := m(r,v) + N(r,v,r_0) + m(r_0,-v), \quad r > r_0\,,$$

where r_0 is an arbitrary fixed positive number (one may as well take $r_0 = 1$). Note that (if it does not cause any misunderstanding) we will write $T(r,v)$ instead of $T(r,v,r_0)$ and so on.

Let $\lambda_k(r) = \lambda_k(\overline{C(0,r)})$. Recall the Carleman's formula in Grishin's notation:

$$\frac{1}{r^k}\int_0^\pi v(re^{i\varphi})\sin k\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\, dt + \frac{1}{r_0^k}\int_0^\pi v(r_0e^{i\varphi})\sin k\varphi\, d\varphi\,,$$

where

$$d\lambda_k(\tau e^{i\varphi}) = \frac{\sin k\varphi}{\sin\varphi}\tau^{k-1}\, d\lambda(\tau e^{i\varphi}).$$

The function $\sin k\varphi/\sin\varphi$ is defined for $\varphi = 0,\, \pi$. In particular for $k = 1$

$$\frac{1}{r}\int_0^\pi v(re^{i\varphi})\sin\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda(t)}{t^3}\, dt + \frac{1}{r_0}\int_0^\pi v(r_0e^{i\varphi})\sin\varphi\, d\varphi\,. \tag{2}$$

The formula (2) can be written as

$$T(r,v) = T(r,-v)\,. \tag{3}$$

Definition 2.1. A strictly positive continuous increasing unbounded function $\gamma(r)$, which is defined on the half-axis $[0,+\infty)$ is called *a growth function*.

Definition 2.2. The following values

$$\beta[\gamma] = \limsup_{r\to\infty}\frac{\ln\gamma(r)}{\ln r}, \quad \alpha[\gamma] = \liminf_{r\to\infty}\frac{\ln\gamma(r)}{\ln r}$$

are called *an order and lower order* of the growth function γ, respectively.

Definition 2.3. The values $\beta[rT(r,v)]$ and $\alpha[rT(r,v)]$ are called *an order and lower order* of the function $v \in J\delta$, respectively.

3. Fourier coefficients of delta-subharmonic functions

The Fourier coefficients of a function $v \in J\delta$ are defined by the formula

$$c_k(r,v) = \frac{2}{\pi}\int_0^\pi v(re^{i\theta})\sin k\theta\, d\theta, \quad k \in \mathbb{N}\,.$$

Let λ be a full measure of $v \in J\delta$, then

$$c_k(r,v) = \alpha_k r^k + \frac{2r^k}{\pi}\int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\, dt, \quad k \in \mathbb{N}\,, \tag{4}$$

where $\alpha_k = r_0^{-k}c_k(r_0,v)$, and

$$\begin{gathered} c_k(r,v) = \alpha_k r^k + \frac{r^k}{\pi k r_0^{2k}}\iint_{\overline{C_+(0,r_0)}}\frac{\sin k\varphi}{\Im\zeta}\tau^k\, d\lambda(\zeta) + \\ \frac{r^k}{\pi k}\iint_{D_+(r_0,r)}\frac{\sin k\varphi}{\tau^k\Im\zeta}\, d\lambda(\zeta) - \frac{1}{r^k\pi k}\iint_{\overline{C_+(0,r)}}\frac{\sin k\varphi}{\Im\zeta}\tau^k\, d\lambda(\zeta)\,, \end{gathered} \tag{5}$$

where $\zeta = \tau e^{i\varphi}$ [**3**]. From definition of $c_k(r,v)$, below inequality follows:

$$|c_k(r,v)| \le \frac{2k}{\pi}\int_0^\pi |v(re^{i\varphi})|\sin\varphi\, d\varphi\,.$$

By this inequality and (3) we obtain

$$rT(r,v) \ge \frac{\pi}{2k}|c_k(r,v)|, \quad k = 1,2,\dots\,. \tag{6}$$

4. The functions with the full measure on the finite system of rays

The main result of this paper is the following theorem.

Theorem 4.1. *If $v \in SK$ is the subharmonic function in $\mathbb{C}_+$ of infinite order with the full measure λ on the finite system of rays $\mathbb{L}_k = \{z : \arg z = e^{i\theta_k},\ \theta_k = \frac{\pi}{2^k}\}$, $k = 1, \dots, q$, then its lower order equals to infinity.*

Proof. Let us assume that $0 \notin \operatorname{supp} v$. As λ lies on the finite system of rays that by formulae (5) for Fourier coefficients of the function v we obtain

$$c_n(r,v) = \alpha_n r^n + \sum_{k=1}^{q} \frac{r^n}{\pi n r_0^{2n}} \sin\theta_k n \int_0^{r_0} t^{n-1} d\lambda(t)$$
$$+ \sum_{k=1}^{q} \frac{r^n \sin\theta_k n}{\pi n} \int_{r_0}^{r} \frac{d\lambda(t)}{t^{n+1}} - \sum_{k=1}^{q} \frac{1}{r^n \pi n} \sin\theta_k n \int_0^{r} t^{n-1} d\lambda(t),\ n = 1, 2, \dots .$$

Assume r_0 is so that $C(0, r_0) \notin \operatorname{supp} v$. Then we obtain

$$c_n(r,v) = \alpha_n r^n + \sum_{k=1}^{q} \frac{1}{\pi n} \sin\theta_k n \int_{r_0}^{r} \Big[\frac{1}{t}\Big(\frac{r}{t}\Big)^n - \frac{1}{t}\Big(\frac{t}{r}\Big)^n\Big] d\lambda(t)\,. \tag{7}$$

Applying integration by parts twice in (7), we get

$$\begin{aligned} c_n(r,v) = \alpha_n r^n + \frac{2}{\pi}\tilde{N}(r) \sum_{k=1}^{q} \sin\theta_k n + \frac{(n+1) r^n}{\pi} \sum_{k=1}^{q} \sin\theta_k n \times \\ \int_{r_0}^{r} \frac{\tilde{N}(r)}{t^{n+1}}\, dt - \frac{n-1}{\pi r^n} \sum_{k=1}^{q} \sin\theta_k n \int_{r_0}^{r} t^{n-1} \tilde{N}(r) dt\,, \end{aligned} \tag{8}$$

where $\tilde{N}(r) = \displaystyle\int_{r_0}^{r} \frac{\lambda(t)}{t^2} dt$.

As

$$\frac{n-1}{\pi r^n} \int_{r_0}^{r} t^{n-1} \tilde{N}(r) dt \le \frac{\tilde{N}(r)(n-1)}{\pi r^n} \int_0^{r} t^{n-1} dt = \frac{\tilde{N}(r)}{\pi}\,,$$

that from (8) with $n = 2^{q-1} + 2^{q+1} l$, $l = 1, 2, \dots$, we obtain

$$\frac{c_n(r,v)}{r^n} \ge \alpha_n + \frac{n+1}{\pi} \int_{r_0}^{r} \frac{\tilde{N}(r)}{t^{n+1}}\, dt + \frac{\tilde{N}(r)}{\pi r^n}\,. \tag{9}$$

If the function $\tilde{N}(r)$ has infinite order then the integral which stands in the right part of the last inequality is unlimited as $r \to \infty$ because

$$\int_r^{\infty} \frac{\tilde{N}(t)}{t^{n+1}}\, dt \ge \frac{\tilde{N}(r)}{n r^n}, \quad n = 1, 2, \dots ,$$

and right part of this inequality can be made as much as big for a suitable choice of r. By this inequality and (6), from (9) we get the desired statement.

If $\tilde{N}(r)$ have finite order then there exist positive numbers $K > 0$ and $\rho > 0$ such that $\tilde{N}(r) \le K r^{\rho}$ for all $r > 0$. It is possible to consider that ρ is not an integer. From here it follows that

$$K2^{\rho}r^{\rho} \geq \tilde{N}(2r) \geq \int_r^{2r} \frac{\lambda(t)}{t^2}\,dt \geq \lambda(r)\int_r^{2r} \frac{dt}{t^2} = \frac{\lambda(r)}{2r}\,,$$

i.e.

$$\lambda(r) \leq K2^{\rho+1}r^{\rho+1}\,.$$

In this case from the paper [**2**] it follows that there exists a function $g \in JS$ of order ρ with full measure λ. Then the function $G = v - g \in J\delta$ and $\lambda_G \equiv 0$.
Further we need a lemma.

Lemma 4.2. *If $G \in J\delta$ and $\lambda_G \equiv 0$, then $G(z) = \Im f(z)$, where $f(z)$ is an entire real function.*

Proof. Remind that the entire function is called real if $f(\mathbb{R}) \subset \mathbb{R}$ [**4**].
As the full measure of function G equals to zero then from (1) it follows that for any $R > 0$

$$G(z) = \frac{R}{2\pi}\int_0^{\pi} \frac{\partial G(z, Re^{i\varphi})}{\partial n}\, G(Re^{i\varphi})\,d\varphi\,, z \in C_+(0,R)\,.$$

The right part is a harmonic function in a half-disc $C_+(0,R)$, which is extended to zero on an interval $(-R,R)$ continuously. As R is an arbitrary positive number, then function $G(z)$ is harmonic in a half-plane $\mathbb{C}_+$, which is extended to zero on real axis, continuously. By the principle of symmetry this function is extended as harmonic on the lower half-plane. Then there exists harmonic function $h(z)$ on the complex plane such that $f(\mathbb{R}) = 0$ and $G(z) = h(z)$ for $\Im z > 0$.
Let $-h_1(z)$ be a function which is harmoniously conjugated to function $h(z)$. Then $f(z) = h_1(z) + ih(z)$ is an entire function, real on a real axis and $h(z) = \Im f(z)$. The lemma is proved. □

According to lemma we have $G(z) = \Im f(z)$, where $f(z)$ is an entire real function,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n\,.$$

If we have $a_n \neq 0$ for only finite number, then $f(z)$ is a polynomial, hence G and v have finite orders that contradicts to the condition.
As

$$c_n(r,G) = a_n r^n\,, n = 1,2,\ldots\,,$$

then from the inequality we have

$$rT(r,v) \geq rT(r,G) - rT(r,g) \geq \frac{\pi}{2n}\left|c_n(r,G)\right| + O(r^{\rho}) \geq$$
$$\frac{1}{2}\left|a_n\right| r^n + O(r^{\rho})\,, r \to \infty\,, n = 1,2,\ldots\,.$$

It follows that $\alpha[rT(r,v)] = \infty$. The theorem is proved. □

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