

## RESULTS ON CERTAIN NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

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ABSTRACT. In the paper we shall mainly concern about the special types of non-linear differential polynomial sharing a small function as introduced in [20]. Our main result will improve, unify and generalize a number of recent results.

### 1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let  $f$  and  $g$  be two non-constant meromorphic functions and let  $a$  be a finite complex number. We say that  $f$  and  $g$  share  $a$  CM(counting multiplicities), provided that  $f - a$  and  $g - a$  have the same zeros with the same multiplicities. Similarly, we say that  $f$  and  $g$  share  $a$  IM(ignore multiplicities), provided that  $f - a$  and  $g - a$  have the same zeros ignoring multiplicities. In addition we say that  $f$  and  $g$  share  $\infty$  CM, if  $1/f$  and  $1/g$  share  $0$  CM, and we say that  $f$  and  $g$  share  $\infty$  IM, if  $1/f$  and  $1/g$  share  $0$  IM.

Let  $m$  be a positive integer or infinity and  $a \in \mathbb{C} \cup \{\infty\}$ . We denote by  $E_m(a; f)$  the set of all  $a$ -points of  $f$  with multiplicities not exceeding  $m$ , where an  $a$ -point is counted according to its multiplicity. Also we denote by  $\overline{E}_m(a; f)$  the set of distinct  $a$ -points of  $f(z)$  with multiplicities not greater than  $m$ . If  $\alpha$  is a small function we define that  $E_m(\alpha, f) = E_m(\alpha, g)$  ( $\overline{E}_m(\alpha, f) = \overline{E}_m(\alpha, g)$ ), which means  $E_m(0, f - \alpha) = E_m(0, g - \alpha)$  ( $\overline{E}_m(0, f - \alpha) = \overline{E}_m(0, g - \alpha)$ ).

If for some  $a \in \mathbb{C} \cup \{\infty\}$ ,  $E_m(a, f) = E_m(a, g)$  ( $\overline{E}_m(a, f) = \overline{E}_m(a, g)$ ) holds for  $m = \infty$  we say that  $f, g$  share the value  $a$  CM (IM).

We adopt the standard notations of value distribution theory (see [6]). We denote by  $T(r)$  the maximum of  $T(r, f)$  and  $T(r, g)$ . The notation  $S(r)$  denotes any quantity satisfying  $S(r) = o(T(r))$  as  $r \rightarrow \infty$ , outside of a possible exceptional set of finite linear measure.

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Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where  $a$  is a value in the extended complex plane.

Yang and Hua [22] made some vital contribution by showing that conclusions similar to the four value theorem can be obtained when two specific types of non-linear differential polynomials namely differential monomials share the same value. Below we state their results.

**Theorem A.** [22] *Let  $f$  and  $g$  be two non-constant meromorphic functions,  $n \geq 11$  be a positive integer and  $a \in \mathbb{C} - \{0\}$ . If  $f^n f'$  and  $g^n g'$  share a CM, then either  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c_1, c_2$  and  $c$  are three constants satisfying  $(c_1 c_2)^{n+1} c^2 = -1$  or  $f \equiv tg$  for a constant  $t$  such that  $t^{n+1} = 1$ .*

This result may be considered as the inception of new era in the direction of value sharing of differential polynomials and the uniqueness of its generating meromorphic function. The introduction of the new notion of scaling between CM and IM, known as weighted sharing of values by I. Lahiri [7]-[8] in 2001 (for further details see [1]-[4], [11]-[15], [17]-[18]).

Lin and Yi [16] improved the result of Fang and Hong [5] in the following manner.

**Theorem B.** [16] *Let  $f$  and  $g$  be two non-constant meromorphic functions satisfying  $\Theta(\infty, f) > \frac{2}{(n+1)}$ ,  $n(\geq 12)$  an integer. If  $f^n(f-1)f'$  and  $g^n(g-1)g'$  share 1 CM, then  $f \equiv g$ .*

**Theorem C.** [16] *Let  $f$  and  $g$  be two non-constant meromorphic functions and  $n(\geq 13)$  be an integer. If  $f^n(f-1)^2 f'$  and  $g^n(g-1)^2 g'$  share 1 CM, then  $f \equiv g$ .*

In 2005, Xiong, Lin and Mori [21] considered a new type of non-linear differential polynomial. Suppose  $h$  is a non-constant meromorphic function and  $\psi_1(h) = h^{n+1}(g^m + a) + \alpha$ , where  $a$  is a constant and  $\alpha \not\equiv 0, \infty$  is a small function. They proved the following theorem.

**Theorem D.** [21] *Let  $f$  and  $g$  be two transcendental meromorphic functions. Let  $m, n, k$  are positive integers such that  $(k-1)n > 14 + 3m + k(10+m)$  and  $E_k(0, \psi_1'(f)) = E_k(0, \psi_1'(g))$ , then*

(i) *if  $m \geq 2$ , then  $f(z) = g(z)$ ;*

(ii) *if  $m = 1$ , then either  $f(z) \equiv g(z)$ , or  $f$  and  $g$  satisfy the algebraic equation  $R(f, g) \equiv 0$ , where  $R(\omega_1, \omega_2) = (n+1)(\omega_1^{n+2} - \omega_2^{n+2}) - (n+2)(\omega_1^{n+1} - \omega_2^{n+1})$ .*

In 2007, Shen and Li [20] improved and supplemented Theorem D. In 2008, C. Meng [18], improved and supplemented Theorem D by the notion of weighted sharing. Here we mention the following theorem of Meng.

**Theorem E.** [18] *Let  $f$  and  $g$  be two transcendental meromorphic functions. Let  $\psi'_1(f)$  and  $\psi'_1(g)$  share 0 IM. If  $n > 4m + 22$  then the conclusion of Theorem D holds.*

Throughout the paper we define two non-zero polynomials  $P_1(z)$  and  $P(z)$  as follows:

$$(1) \quad P_1(z) = \frac{a_m}{n+m+1}z^m + \frac{a_{m-1}}{n+m}z^{m-1} + \dots + \frac{a_0}{n+1},$$

and

$$(2) \quad P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0,$$

where  $m \geq 1$  is an integer and  $a_0, a_1, \dots, a_m$  are complex constants.

Let  $P(z)$  be non-constant and  $a_m \neq 0, a_0 \neq 0$ . Let  $t$  be the number of distinct roots of the equation  $P(z) = 0$ . We define  $s$  by

$$(3) \quad s = \frac{4m}{t} - (m - 1).$$

Clearly  $t \leq m$ .

Next we recall the following result of Zhang, Chen and Lin [26] since it has some relevance with the above discussion.

**Theorem F.** [26] *Let  $f$  and  $g$  be two non-constant meromorphic functions. Let  $n$  and  $m$  be two positive integers such that  $n > \max\{m + 10, 3m + 3\}$  and  $P(z)$  be such that  $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$  are complex constants. If  $f^n P(f)f'$  and  $g^n P(g)g'$  share  $(1, \infty)$  then either  $f(z) = tg(z)$  for a constant  $t$  such that  $t^d = 1$ , where  $d = \gcd(n + m + 1, \dots, n + m + 1 - i, \dots, n + 1)$ ,  $a_{m-i} \neq 0$  for some  $i \in \{0, 1, 2, \dots, m\}$  or  $f$  and  $g$  satisfy the algebraic equation  $R(f, g) \equiv 0$ , where*

$$R(\omega_1, \omega_2) = \omega_1^{n+1} \left( \frac{a_m \omega_1^m}{n+m+1} + \frac{a_{m-1} \omega_1^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) - \omega_2^{n+1} \left( \frac{a_m \omega_2^m}{n+m+1} + \frac{a_{m-1} \omega_2^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right).$$

In 2011, Zhang-Xu [27] generalized Theorem F for small functions. Recently, Sahoo and Seikh [19] improved Theorem F in the following way.

**Theorem G.** [26] *Let  $f$  and  $g$  be two non-constant meromorphic functions. Let  $n, m, k$  be three positive integers such that  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$  and  $P(z)$  be such that  $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$  are complex constants. If  $E_k(\alpha, f^n P(f)f') = E_k(\alpha, g^n P(g)g')$  and one of the following holds:*

- (i)  $k \geq 3$  and  $\Theta(\infty; f) > 0, \Theta(\infty; g) > 0$  and  $n > \max\{3m + 1, 3m + 9\}$ ;
- (ii)  $k = 2$  and  $n > \max\{3m + 1, \frac{3m}{2} + 12\}$ ;
- (iii)  $k = 1$  and  $n > 3m + 17$ ,

then the conclusion of Theorem F holds.

**Remark.** It should be noted that in Theorem G the condition  $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$  is only required when  $m = 1$ . Otherwise this condition is redundant.

Let  $m^*$  be a non-negative integer defined as follows:

$$m^* = \begin{cases} m, & \text{if } a_m \neq 0 \\ 0, & \text{if } a_0 \neq 0 \text{ and } a_i = 0, 1 \leq i \leq m \end{cases} .$$

For a non-constant meromorphic function  $h$  we define  $\psi(h)$  as

$$\psi(h) = \left[ h^{n+1} \left\{ \frac{a_m}{n+m+1} h^m + \frac{a_{m-1}}{n+m} h^{m-1} + \dots + \frac{a_0}{n+1} \right\} \right] + \alpha.$$

In the context of the result of Xiong, Lin and Mori [20] it will be interesting to investigate the conclusions of Theorem G for all possible forms of  $P(z)$  so that all results except Theorem E can be brought under a single umbrella. In this paper we shall obtain two results; one of them improves Theorem E and the other improves all the remaining results to a large extent.

**Theorem 1.1.** Let  $f$  and  $g$  be two non-constant meromorphic functions, and  $\alpha(z) (\neq 0, \infty)$  be a small function with respect to  $f$  and  $g$ . Also we suppose that  $E_k(0, \psi'(f)) = E_k(0, \psi'(g))$ . If

- (a)  $k \geq 3$  and  $n > \max\{m^* + 10, s\}$ ;
- (b)  $k = 2$  and  $n > \max\{\frac{3m^*}{2} + 12, s\}$ ;
- (c)  $k = 1$  and  $n > \max\{3m^* + 18, s\}$ ,

then the following conclusions hold.

- (I) When  $a_m \neq 0, a_0 \neq 0$  and at least one of  $a_{m-i} \neq 0, i = 1, 2, \dots, m-1$  then one of the following two conditions holds:
  - (I1)  $f(z) \equiv tg(z)$  for a constant  $t$  such that  $t^d = 1$ , where  $d = \gcd(n+m+1, n+m, \dots, n+m+1-i, \dots, n+1)$ ,  $a_{m-i} \neq 0$  for some  $i = 0, 1, 2, \dots, m$ ;
  - (I2)  $f$  and  $g$  satisfy the algebraic equation  $R(f, g) \equiv 0$ , where
 
$$R(\omega_1, \omega_2) = \omega_1^{n+1}(a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \dots + a_0) - \omega_2^{n+1}(a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \dots + a_0),$$
- (II) When  $a_m \neq 0, a_0 \neq 0$  and all of  $a_{m-i}$ 's,  $i = 1, 2, \dots, m-1$  are zero then
  - (II1) if  $m = 1$ ,  $\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$ ; or
  - (II2) if  $m \geq 2$ , we have  $f \equiv tg$  for some constant  $t$ , satisfying  $t^d \equiv 1$ , where  $d = \gcd(m, n+1)$ ,
- (III) When  $|a_m| + |a_0| \neq 0$ , but  $|a_m| \cdot |a_0| = 0$  and all of  $a_{m-i}$ 's,  $i = 1, 2, \dots, m-1$  are zero then one of the following two conditions holds:
  - (III1)  $f(z) \equiv tg(z)$  where  $t$  is a constant satisfying  $t^{n+m^*+1} = 1$ ;

- (III2)  $a_{m^*}^2 [f^{n+m^*+1}]' [g^{n+m^*+1}]' \equiv \alpha^2$ . In particular when  $\alpha(z) = d = \text{constant}$ , we get  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$ , where  $c_1$ ,  $c_2$  and  $c$  are constants satisfying
- $$a_{m^*}^2 (c_1 c_2)^{n+m^*+1} ((n+m^*+1)c)^2 = -d^2.$$

**Theorem 1.2.** *Let  $f$  and  $g$  be two non-constant meromorphic functions, and  $\alpha(z) (\neq 0, \infty)$  be a small function with respect to  $f$  and  $g$ . Also we suppose that  $\overline{E}_k(0, \psi'(f)) = \overline{E}_k(0, \psi'(g))$ , where  $n > \max\{4m^* + 22, s\}$  is an integer. Then the conclusions of Theorem 1.1 hold.*

**Remark.** *In the above theorem when  $k \rightarrow \infty$ , we get a generalized version of Theorem E.*

We now explain following definitions and notations which are used in the paper.

**Definition 1.3.** [13] *Let  $p$  be a positive integer and  $a \in \mathbb{C} \cup \{\infty\}$ .*

- (i)  $N(r, a; f | \geq p)$  ( $\overline{N}(r, a; f | \geq p)$ ) denotes the counting function (reduced counting function) of those  $a$ -points of  $f$  whose multiplicities are not less than  $p$ .
- (ii)  $N(r, a; f | \leq p)$  ( $\overline{N}(r, a; f | \leq p)$ ) denotes the counting function (reduced counting function) of those  $a$ -points of  $f$  whose multiplicities are not greater than  $p$ .

**Definition 1.4.** [23] *For  $a \in \mathbb{C} \cup \{\infty\}$  and a positive integer  $p$  we denote by  $N_p(r, a; f)$  the sum  $\overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq p)$ . Clearly  $N_1(r, a; f) = \overline{N}(r, a; f)$ .*

**Definition 1.5.** *Let  $k$  be a positive integer and for  $a \in \mathbb{C} - \{0\}$ ,  $\overline{E}_k(a; f) = \overline{E}_k(a; g)$ . Let  $z_0$  be a zero of  $f(z) - a$  of multiplicity  $p$  and a zero of  $g(z) - a$  of multiplicity  $q$ . We denote by  $\overline{N}_L(r, a; f)$  the counting function of those  $a$ -points of  $f$  and  $g$  where  $p > q \geq 1$ , by  $\overline{N}_{f>s}(r, a; g)$  ( $\overline{N}_{g>s}(r, a; f)$ ) the counting functions of those  $a$ -points of  $f$  and  $g$  for which  $p > q = s$  ( $q > p = s$ ), by  $N_E^1(r, a; f)$  the counting function of those  $a$ -points of  $f$  and  $g$  where  $p = q = 1$  and by  $\overline{N}_E^{(2)}(r, a; f)$  the counting function of those  $a$ -points of  $f$  and  $g$  where  $p = q \geq 2$ , each point in these counting functions is counted only once. Similarly, we can define  $\overline{N}_L(r, a; g)$ ,  $N_E^1(r, a; g)$ , and  $\overline{N}_E^{(2)}(r, a; g)$ . We denote by  $\overline{N}_{f \geq k+1}(r, a; f | g \neq a)$  ( $\overline{N}_{g \geq k+1}(r, a; g | f \neq a)$ ) the reduced counting functions of those  $a$ -points of  $f$  and  $g$  for which  $p \geq k+1$  and  $q = 0$  ( $q \geq k+1$  and  $p = 0$ ).*

**Definition 1.6.** [9] *Let  $a, b \in \mathbb{C} \cup \{\infty\}$ . We denote by  $N(r, a; f | g \neq b)$  the counting function of those  $a$ -points of  $f$ , counted according to multiplicity, which are not the  $b$ -points of  $g$ .*

## 2. LEMMAS

Let  $F$  and  $G$  be two non-constant meromorphic functions defined in  $\mathbb{C}$ . We define the function  $H$  as:

$$(4) \quad H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

**Lemma 2.1.** [13] *Let  $f$  be a non-constant meromorphic function and let  $a_n(z) (\neq 0)$ ,  $a_{n-1}(z)$ , ...,  $a_0(z)$  be meromorphic functions such that  $T(r, a_i(z)) = S(r, f)$  for  $i = 0, 1, 2, \dots, n$ . Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

**Lemma 2.2.** [28] *Let  $f$  be a non-constant meromorphic function, and  $p, k$  be positive integers. Then*

$$(5) \quad N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$

$$(6) \quad N_p(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

**Lemma 2.3.** [10] *If  $N(r, 0; f^{(k)} \mid f \neq 0)$  denotes the counting function of those zeros of  $f^{(k)}$  which are not the zeros of  $f$ , where a zero of  $f^{(k)}$  is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\bar{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\bar{N}(r, 0; f \mid \geq k) + S(r, f).$$

**Lemma 2.4.** [22] *Let  $f$  and  $g$  be two non-constant meromorphic functions,  $n \geq 6$  be an integer. If  $f^n f' g^n g' = 1$  then  $f(z) = c_1 e^{cz}$ ,  $g(z) = c_2 e^{-cz}$  where  $c, c_1$  and  $c_2$  are constants satisfying  $(c_1 c_2)^{n+1} c^2 = -1$ .*

**Lemma 2.5.** *Let  $f, g$  be two non-constant meromorphic functions and  $n$  be a positive integer such that  $n > 6$ . If  $a_{m^*}^2 (f^{n+m^*+1})' (g^{n+m^*+1})' \equiv d^2$ , then  $f = c_1 e^{cz}$ ,  $g = c_2 e^{-cz}$ , where  $c_1, c_2$  and  $c$  are constants such that  $a_{m^*}^2 (c_1 c_2)^{n+m^*+1} ((n+m^*+1)c)^2 = -d^2$ .*

*Proof.* From the given condition we can write

$$(7) \quad f^{n+m^*} f' g^{n+m^*} g' \equiv \left( \frac{d}{a_{m^*} (n+m^*+1)^2} \right)^2 = k^2,$$

where  $k = d/(a_{m^*} (n+m^*+1)^2)$ . We put  $f_1 = \frac{f}{k^{\frac{1}{n+m^*+1}}}$ ,  $g_1 = \frac{g}{k^{\frac{1}{n+m^*+1}}}$ . Then (7) reduces to

$$f_1^{n+m^*} f_1' g_1^{n+m^*} g_1' = 1.$$

Using Lemma 2.4, we have  $f = c_1 e^{cz}$ ,  $g = c_2 e^{-cz}$ , where  $c_1, c_2$  and  $c$  are constants such that  $a_{m^*}^2 (c_1 c_2)^{n+m^*+1} ((n+m^*+1)c)^2 = -d^2$ .  $\square$

**Lemma 2.6.** *Let  $f, g$  be two non-constant meromorphic functions and  $F = \frac{[f^{n+1}P_1(f)]'}{-\alpha}, G = \frac{[g^{n+1}P_1(g)]'}{-\alpha}$ , where  $\alpha(z) (\neq 0, \infty)$  be a small function with respect to  $f$  and  $g$ ,  $n$  is a positive integer such that  $n > m^* + 5$ . If  $H \equiv 0$  then either  $[f^{n+1}P_1(f)]' \equiv [g^{n+1}P_1(g)]'$  or  $[f^{n+1}P_1(f)]'[g^{n+1}P_1(g)]' \equiv \alpha^2$ .*

*Proof.* Since  $H \equiv 0$ , by integration we get

$$(8) \quad \frac{1}{F-1} \equiv \frac{bG+a-b}{G-1},$$

where  $a, b$  are constants and  $a \neq 0$ . We now consider the following cases.

**Case 1.** Let  $b \neq 0$  and  $a \neq b$ .

If  $b = -1$ , then from (8) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore

$$\bar{N}(r, a+1; G) = \bar{N}(r, \infty; F) = \bar{N}(r, \infty; f).$$

So by Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned} (n+m^*+1)T(r, g) &\leq T(r, G) + N_2(r, 0; g^{n+1}P_1(g)) - \bar{N}(r, 0; G) \\ &\leq \bar{N}(r, \infty; G) + \bar{N}(r, 0; G) + \bar{N}(r, a+1; G) + N_2(r, 0; g^{n+1}P(g)) \\ &\quad - \bar{N}(r, 0; G) + S(r, g) \\ &\leq \bar{N}(r, \infty; g) + N_2(r, 0; g^{n+1}P(g)) + \bar{N}(r, \infty; f) + S(r, g) \\ &\leq T(r, f) + (m^*+3)T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set  $I$  with infinite measure such that  $T(r, f) \leq T(r, g)$  for  $r \in I$ .

So for  $r \in I$  we have

$$(n-3)T(r, g) \leq S(r, g),$$

which is a contradiction.

If  $b \neq -1$ , from (8) we obtain that

$$F - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2 \left[G + \frac{a-b}{b}\right]}.$$

So

$$\bar{N}\left(r, \frac{(b-a)}{b}; G\right) = \bar{N}(r, \infty; F) = \bar{N}(r, \infty; f).$$

Using Lemma 2.2 and by the same argument as used in the case when  $b = -1$  we can get a contradiction.

**Case 2.** Let  $b \neq 0$  and  $a = b$ .

If  $b = -1$ , then from (8) we have

$$FG \equiv \alpha^2,$$

that is

$$[f^{n+1}P_1(f)]' [g^{n+1}P_1(g)]' \equiv \alpha^2.$$

If  $b \neq -1$ , from (8) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore

$$\overline{N}\left(r, \frac{1}{1+b}; G\right) = \overline{N}(r, 0; F).$$

So by Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned} (n + m^* + 1) T(r, g) &\leq T(r, G) + N_2(r, 0; g^{n+1}P_1(g)) - \overline{N}(r, 0; G) + S(r, g) \\ &\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+b}; G\right) + N_2(r, 0; g^{n+1}P_1(g)) \\ &\quad - \overline{N}(r, 0; G) + S(r, g) \\ &\leq (m^* + 3)T(r, g) + \overline{N}(r, 0; F) + S(r, g) \\ &\leq \overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + (m^* + 3)T(r, g) \\ &\leq (m^* + 3)\{T(r, g) + T(r, f)\} + S(r, f) + S(r, g). \end{aligned}$$

So for  $r \in I$  we have

$$\{n - m^* - 5\} T(r, g) \leq S(r, g),$$

which is a contradiction since  $n > m^* + 5$ .

**Case 3.** Let  $b = 0$ . From (8) we obtain

$$(9) \quad F \equiv \frac{G + a - 1}{a}.$$

If  $a \neq 1$  then from (9) we get

$$\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore  $a = 1$  and from (9) we obtain

$$F \equiv G.$$

i.e.,

$$[f^{n+1}P_1(f)]' \equiv [g^{n+1}P_1(g)]'.$$

□



**Lemma 2.7.** *Let  $f$  and  $g$  be two non-constant meromorphic functions and  $\alpha(z) (\neq 0, \infty)$  be a small function of  $f$  and  $g$ . Let  $n$  be a positive integer such that  $n > s$ , where  $s$  is defined by (3). Suppose that  $P(z) \neq a_i z^i$ , for  $i = 1, 2, \dots, m$  be a non-constant polynomial. Then*

$$f^n P(f) f' g^n P(g) g' \neq \alpha^2.$$

*Proof.* First suppose that

$$(10) \quad f^n P(f) f' g^n P(g) g' \equiv \alpha^2(z).$$

Let  $d_i$  be the distinct zeros of  $P(z) = 0$  with multiplicity  $p_i$ , where  $i = 1, 2, \dots, t$ ,  $1 \leq t \leq m$  and  $\sum_{i=1}^t p_i = m$ .

Now by the second fundamental theorem for  $f$  and  $g$  we get respectively

$$(11) \quad tT(r, f) \leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + \sum_{i=1}^t \bar{N}(r, d_i; f) - \bar{N}_0(r, 0; f') + S(r, f),$$

and

$$(12) \quad tT(r, g) \leq \bar{N}(r, 0; g) + \bar{N}(r, \infty; g) + \sum_{i=1}^t \bar{N}(r, d_i; g) - \bar{N}(r, 0; g') + S(r, g),$$

where  $\bar{N}(r, 0; f')$  denotes the reduced counting function of those zeros of  $f'$  which are not the zeros of  $f$  and  $f - d_i$ ,  $i = 1, 2, \dots, t$ .  $\bar{N}(r, 0; g')$  can be similarly defined.

Let  $z_0$  be a zero of  $f$  with multiplicity  $p$  but  $\alpha(z_0) \neq 0, \infty$ . Clearly  $z_0$  must be a pole of  $g$  with multiplicity  $q$ . Then from (10) we get  $np + p - 1 = nq + mq + q + 1$ . This gives

$$(13) \quad mq + 2 = (n + 1)(p - q).$$

From (13) we get  $p - q \geq 1$  and so  $q \geq \frac{n-1}{m}$ . Now  $np + p - 1 = nq + mq + q + 1$  gives  $p \geq \frac{n+m-1}{m}$ . Thus we have

$$(14) \quad \bar{N}(r, 0; f) \leq \frac{m}{n+m-1} N(r, 0; f) \leq \frac{m}{n+m-1} T(r, f).$$

Let  $z_1 (\alpha(z_1) \neq 0, \infty)$  be a zero of  $f - d_i$  with multiplicity  $q_i$ ,  $i = 1, 2, \dots, t$ . Then  $z_1$  must be a pole of  $g$  with multiplicity  $r (\geq 1)$ . So from (10) we get  $q_i p_i + q_i - 1 = (n + m + 1)r + 1 \geq n + m + 2$ . This gives  $q_i \geq \frac{n+m+2}{p_i+1}$  for  $i = 1, 2, \dots, t$  and so we get

$$\bar{N}(r, d_i; f) \leq \frac{p_i + 1}{n + m + 3} N(r, d_i; f) \leq \frac{p_i + 1}{n + m + 3} T(r, f).$$

Clearly,

$$(15) \quad \sum_{i=1}^t \bar{N}(r, d_i; f) \leq \frac{m+t}{n+m+3} T(r, f).$$

Similarly, we have

$$(16) \quad \bar{N}(r, 0; g) \leq \frac{m}{n+m-1} T(r, g),$$

and

$$(17) \quad \sum_{i=1}^t \bar{N}(r, d_i; g) \leq \frac{m+t}{n+m+3} T(r, g).$$

Also it is clear from (16) and (17) that

$$(18) \quad \begin{aligned} \bar{N}(r, \infty; f) &\leq \bar{N}(r, 0; g) + \sum_{i=1}^t \bar{N}(r, d_i; g) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g) \\ &\leq \left( \frac{m}{n+m-1} + \frac{m+t}{n+m+3} \right) T(r, g) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g). \end{aligned}$$

Then by (11), (14), (15) and (18) we get

$$(19) \quad \begin{aligned} t T(r, f) &\leq \left( \frac{m}{n+m-1} + \frac{m+t}{n+m+3} \right) \{T(r, f) + T(r, g)\} + \bar{N}_0(r, 0; g') \\ &\quad - \bar{N}_0(r, 0; f') + S(r, f) + S(r, g). \end{aligned}$$

Similarly, we have

$$(20) \quad \begin{aligned} t T(r, g) &\leq \left( \frac{m}{n+m-1} + \frac{m+t}{n+m+3} \right) \{T(r, f) + T(r, g)\} + \bar{N}_0(r, 0; f') \\ &\quad - \bar{N}_0(r, 0; g') + S(r, f) + S(r, g). \end{aligned}$$

So from (19) and (20) we get

$$t\{T(r, f) + T(r, g)\} \leq 2 \left( \frac{m}{n+m-1} + \frac{m+t}{n+m+3} \right) \{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),$$

i.e.,

$$(21) \quad \left( t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} \right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).$$

Since

$$\left( t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} \right) = \frac{(n+m-1)^2 t + 2(n+m-1)(t-2m) - 8m}{(n+m-1)(n+m+3)},$$

we note that when  $n+m-1 > \frac{4m}{t}$ , i.e., when  $n > \frac{4m}{t} - (m-1) = s$ , we have clearly  $t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} > 0$  and so (21) leads to a contradiction. This completes the proof.  $\square$

**Lemma 2.8.** *Let  $f$  and  $g$  be two non-constant meromorphic (entire) functions and  $n(\geq 2)$ ,  $m(\geq 1)$  be two distinct integers satisfying  $n+m \geq d+6$  ( $n+m \geq d+2$ ). Then for two constants  $\lambda, \mu$ , with  $|\lambda| + |\mu| \neq 0$ ,*

$$f^{n+1}(\mu f^m + \lambda) \equiv g^{n+1}(\mu g^m + \lambda)$$

implies the following.

(i) if  $\lambda\mu \neq 0$  and

(a)  $m = 1$ ,  $\Theta(\infty, f) + \Theta(\infty, g) > 4/n + 1$ ; or

(b)  $m \geq 2$  and for some constant  $t$ , satisfying  $t^d \equiv 1$ ,

we have  $f \equiv tg$ , where  $d = (m, n+1)$ ,

(ii) if  $\lambda\mu = 0$ , then  $f = tg$ , where  $t$  is a constant satisfying  $t^{n+m^*+1} = 1$ .

*Proof.* Let  $m = 1$ . In this case noting that  $d = 1 = (n+2, n+1)$ , proceeding in the same way as done in Lemma 2.6 of [11] we can show when  $\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$ , we have  $f \equiv g$ .

Next suppose  $m \geq 2$ . Let  $f \not\equiv tg$  for a constant  $t$  satisfying  $t^d = 1$ . We put  $h = \frac{f}{g}$ .

Then  $h^d \not\equiv 1$ , i.e.,  $(h - v_0)(h - v_1) \dots (h - v_{d-1}) \not\equiv 0$ , where  $v_k = \exp\left(\frac{2k\pi i}{d}\right)$ ,  $k = 0, 1, 2, \dots, d-1$ . First suppose that  $h$  is constant. Now from the given condition we have

$$\mu g^m (h^{n+m+1} - 1) \equiv -\lambda (h^{n+1} - 1).$$

Since  $\gcd(n+1, m) = d$ , it follows that  $\gcd(n+m+1, n+1) = d$ .

Eliminating the common factors, we end up with

$$ag^m(h - \alpha_1)(h - \alpha_2) \dots (h - \alpha_{n+m+1-d}) \equiv (h - \beta_1)(h - \beta_2) \dots (h - \beta_{n+1-d}),$$

where  $\alpha_i$  and  $\beta_j$  are those zeros of  $h^{n+m+1} - 1$  and  $h^{n+1} - 1$  which are not the zeros of  $h^d - 1$ ,  $i = 1, 2, \dots, n+m+1-d$  and  $j = 1, 2, \dots, n+1-d$ . Also we note that none of the  $\alpha_i$ 's coincide with  $\beta_j$ 's. So if  $h = \alpha_i$  or  $\beta_j$ , then we have either  $(h - \beta_1)(h - \beta_2) \dots (h - \beta_{n-d}) \equiv 0$  or  $g \equiv 0$  and in both case we get a contradiction. Hence we assume neither  $h^{n+m+1} \equiv 1$  nor  $h^{n+1} \equiv 1$ . Therefore we may write

$$(22) \quad g^m = -\frac{\lambda}{\mu} \frac{h^{n+1} - 1}{h^{n+m+1} - 1}.$$

It follows from above that  $g$  is a constant, which is impossible. So  $h$  is non-constant. We observe that since a non-constant meromorphic function cannot have more than two Picard exceptional values  $h$  can take at least  $n+m-d-1$  values among  $u_j = \exp\left(\frac{2j\pi i}{n+m+1}\right)$ , where  $j = 0, 1, 2, \dots, n+m$ . Since  $f^m$  has no simple pole  $h - u_j$

has no simple zero for at least  $n + m - d - 1$  values of  $u_j$ , for  $j = 0, 1, 2, \dots, n + m$  and for these values of  $j$  we have  $\Theta(u_j; h) \geq \frac{1}{2}$ , which leads to a contradiction.

Therefore  $h^d \equiv 1$  i.e.,  $f \equiv tg$  for a constant  $t$  satisfying  $t^d = 1$ , where  $d = \gcd(n+1, m)$ .

**Subcase 2.2:** Let  $\lambda\mu = 0$  but  $|\lambda| + |\mu| \neq 0$ . Then from the given condition we get  $f^{n+m^*+1} \equiv g^{n+m^*+1}$  and so  $f \equiv tg$ , where  $t$  is a constant satisfying  $t^{n+m^*+1} = 1$ .  $\square$

### 3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.2.** Since

$$\psi(f) = \left[ f^{n+1} \left\{ \frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \dots + \frac{a_0}{n+1} \right\} \right] + \alpha = f^{n+1} P_1(f) + \alpha$$

and

$$\psi(g) = \left[ g^{n+1} \left\{ \frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \dots + \frac{a_0}{n+1} \right\} \right] + \alpha = g^{n+1} P_1(g) + \alpha,$$

we have

$$\psi'(f) = f^n [a_m f^m + a_{m-1} f^{m-1} + \dots + a_0] f' + \alpha' = f^n P(f) f' + \alpha',$$

and

$$\psi'(g) = g^n [a_m g^m + a_{m-1} g^{m-1} + \dots + a_0] g' + \alpha' = g^n P(g) g' + \alpha'.$$

Let

$$F_1 = f^{n+1} \left\{ \frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \dots + \frac{a_0}{n+1} \right\} = f^{n+1} P_1(f),$$

$$G_1 = g^{n+1} \left\{ \frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \dots + \frac{a_0}{n+1} \right\} = g^{n+1} P_1(g),$$

$$F = \frac{f^n [a_m f^m + a_{m-1} f^{m-1} + \dots + a_0] f'}{-\alpha'} = \frac{f^n P(f) f'}{-\alpha'} = \frac{(f^{n+1} P_1(f))'}{-\alpha'},$$

and

$$G = \frac{g^n [a_m g^m + a_{m-1} g^{m-1} + \dots + a_0] g'}{-\alpha'} = \frac{g^n P(g) g'}{-\alpha'} = \frac{(g^{n+1} P_1(g))'}{-\alpha'}.$$

Since  $\overline{E}_k(0, \psi'(f)) = \overline{E}_k(0, \psi'(g))$ , it follows that  $\overline{E}_k(1, F) = \overline{E}_k(1, G)$ , except the zeros and poles of  $\alpha'$ . Also  $F'_1 = -\alpha' F$  and  $G'_1 = -\alpha' G$ .

**Case 1.** Let  $H \neq 0$ .

Let  $z_0$  be a simple zero of  $F - 1$ . Then by a simple calculation we see that  $z_0$  is a zero of  $H$  and hence

$$(23) \quad N_E^1(r, 1; F) = N_E^1(r, 1; G) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Also

$$\begin{aligned}
 N(r, \infty; H) &\leq \bar{N}(r, 0; F \mid \geq 2) + \bar{N}(r, 0; G \mid \geq 2) + \bar{N}(r, \infty; F \mid \geq 2) + \bar{N}(r, \infty; G \mid \geq 2) \\
 &\quad + \bar{N}_{F \geq k+1}(r, 1; F \mid G \neq 1) + \bar{N}_{G \geq k+1}(r, 1; G \mid G \neq 1) + \bar{N}_L(r, 1; F) \\
 (24) \quad &\quad + \bar{N}_L(r, 1; G) + \bar{N}_0(r, 0; F') + \bar{N}_0(r, 0; G'),
 \end{aligned}$$

where  $\bar{N}_0(r, 0; F')$  is the reduced counting function of those zeros of  $F'$  which are not the zeros of  $F(F-1)$ .  $\bar{N}_0(r, 0; G')$  is similarly defined.

Using (23), (24) and Lemma 2.2 and noting that

$$\begin{aligned}
 \bar{N}_{G>1}(r, 1; F) + \bar{N}(r, 1; F \mid \geq 2) &= \bar{N}_E^{(2)}(r, 1; F) + \bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) \\
 &\quad + \bar{N}_{F \geq m+1}(r, 1; F \mid G \neq 1) + S(r),
 \end{aligned}$$

by the second fundamental theorem we get

$$\begin{aligned}
 (n + m^* + 1)T(r, f) &\leq T(r, F) + N_3(r, 0; F_1) - N_2(r, 0; F) + S(r, f) \\
 &\leq \bar{N}(r, \infty; f) + \bar{N}(r, 0; F) + N_E^1(r, 1; F) + \bar{N}_{G>1}(r, 1; F) + \bar{N}(r, 1; F \mid \geq 2) \\
 &\quad + N_3(r, 0; F_1) - N_2(r, 0; F) - N_0(r, 0; F') + S(r, f) + S(r, g) \\
 &\leq 2\bar{N}(r, \infty; f) + \bar{N}(r, \infty; g) + \bar{N}(r, 0; G \mid \geq 2) + 2\bar{N}_L(r, 1; F) \\
 &\quad + 2\bar{N}_L(r, 1; G) + 2\bar{N}_{F \geq k+1}(r, 1; F \mid G \neq 1) + \bar{N}_{G \geq k+1}(r, 1; G \mid F \neq 1) \\
 &\quad + \bar{N}_E^{(2)}(r, 1; G) + \bar{N}_0(r, 0; G') + N_3(r, 0; F_1) + S(r).
 \end{aligned}$$

(25)

Using Lemma 2.2 and 2.3 we obtain that

$$\begin{aligned}
 &2\bar{N}_{F \geq k+1}(r, 1; F \mid G \neq 1) + 2\bar{N}_L(r, 1; F) + \bar{N}_E^{(2)}(r, 1; F) \\
 &\leq 2N(r, 0; F' \mid F \neq 0) + S(r, f) \\
 &\leq 2\bar{N}(r, \infty; f) + 2\bar{N}(r, 0; F) + S(r, f) \\
 &\leq 4\bar{N}(r, \infty; f) + 4\bar{N}(r, 0; f) + 2m^*T(r, f) + S(r, f),
 \end{aligned}$$

(26)

and

$$\begin{aligned}
 &\bar{N}(r, 0; G \mid \geq 2) + \bar{N}_{G \geq m+1}(r, 1; G \mid F \neq 1) + 2\bar{N}_L(r, 1; G) + \bar{N}_0(r, 0; G') \\
 &\leq 2\bar{N}(r, 0; G' \mid G \neq 0) + \bar{N}(r, 0; G' \mid G = 0) + S(r) \\
 &\leq N(r, 0; G' \mid G \neq 0) + \bar{N}(r, 0; G') + S(r) \\
 &\leq 2\bar{N}(r, \infty; g) + \bar{N}(r, 0; G) + N_2(r, 0; G) + S(r). \\
 &\leq 4\bar{N}(r, \infty; g) + 5\bar{N}(r, 0; g) + 2m^*T(r, f) + S(r).
 \end{aligned}$$

(27)

Using (26) and (27) in (25) we have

$$\begin{aligned}
(n + m^* + 1)T(r, f) &\leq 6\overline{N}(r, \infty; f) + 5\overline{N}(r, \infty; g) + 7\overline{N}(r, 0; f) + 5\overline{N}(r, 0; g) \\
&\quad + 3m^*T(r, f) + 2m^*T(r, g) + S(r) \\
&\leq (5m^* + 23)T(r) + S(r).
\end{aligned}$$

In a similar way we get

$$(n + m^* + 1)T(r, g) \leq (5m^* + 23)T(r) + S(r).$$

So from the above two inequalities we get

$$(n - 4m^* - 22)T(r) \leq S(r),$$

which is a contradiction.

**Case 2.** Now suppose  $H \equiv 0$ . Then by Lemma 2.6 we see that either  $FG \equiv \alpha^2$  or  $F \equiv G$ . First suppose  $P(z)$  is a non-constant polynomial with  $a_m \neq 0$  and  $a_0 \neq 0$ , then by Lemma 2.7,  $FG \not\equiv \alpha^2$ . Next let  $|a_m| + |a_0| \neq 0$  but  $|a_m| \cdot |a_0| = 0$  and all  $a_{m-i}$ ,  $i = 1, 2, \dots, m-1$  are zero. Then  $FG \equiv \alpha^2$  implies

$$a_{m^*}^2 (f^{n+m^*+1})' (g^{n+m^*+1})' \equiv \alpha^2.$$

In particular, if  $\alpha = d = \text{constant}$ , the conclusion of the theorem follows from Lemma 2.5.

So we must have  $F \equiv G$  i.e.,  $(f^{n+1}P_1(f))' \equiv (g^{n+1}P_1(g))'$ . Integrating, we obtain

$$f^{n+1}P_1(f) \equiv g^{n+1}P_1(g) + c.$$

If possible suppose  $c \neq 0$ .

Now using the second fundamental theorem we get

$$\begin{aligned}
&(n + m^* + 1)T(r, f) \\
&\leq \overline{N}(r, 0; f^{n+1}P_1(f)) + \overline{N}(r, \infty; f^{n+1}P_1(f)) + \overline{N}(r, c_0; f^{n+1}P_1(f)) \\
&\leq \overline{N}(r, 0; f) + m^*T(r, f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; g^{n+1}P_1(g)) \\
&\leq (m^* + 2)T(r, f) + \overline{N}(r, 0; g) + m^*T(r, g) + S(r, f) \\
&\leq (m^* + 2)T(r, f) + (m^* + 1)T(r, g) + S(r, f) + S(r, g) \\
&\leq \{2m^* + 3\}T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n + m^* + 1)T(r, g) \leq \{2m^* + 3\}T(r) + S(r).$$

Combining these we get

$$(n - m^* - 2)T(r) \leq S(r),$$

which is a contradiction since  $n > m^* + 2$ .

Therefore  $c = 0$  and so

$$f^{n+1}P_1(f) \equiv g^{n+1}P_1(g).$$

i.e.,

$$\begin{aligned} \frac{a_m}{n+m+1}f^{n+m+1} + \frac{a_{m-1}}{n+m}f^{n+m} + \dots + \frac{a_0}{n+1}f^{n+1} &\equiv \\ \frac{a_m}{n+m+1}g^{n+m+1} + \frac{a_{m-1}}{n+m}g^{n+m} + \dots + \frac{a_0}{n+1}g^{n+1}. & \end{aligned}$$

If  $a_{m-i} = 0$ , for  $i = 1, 2, \dots, m-1$ , then since  $P(z)$  is a non-zero polynomial, it follows that  $|a_m| + |a_0| \neq 0$ . If  $|a_m| \cdot |a_0| \neq 0$ , the conclusion of the theorem follows from Lemma 2.8(i), otherwise it follows from Lemma 2.8(ii). Let at least one of  $a_{m-i} \neq 0$ , for  $i = 1, 2, \dots, m-1$ . Suppose  $h = \frac{f}{g}$ . If  $h$  is a constant, by putting  $f = hg$  in the above expression we get

$$\begin{aligned} \frac{a_m}{n+m+1}g^m(h^{n+m+1} - 1) + \frac{a_{m-1}}{n+m}g^{m-1}(h^{n+m} - 1) + \dots + \frac{a_1}{n+2}g(h^{n+2} - 1) \\ + \frac{a_0}{n+1}(h^{n+1} - 1) \equiv 0, \end{aligned}$$

which implies that  $h^d = 1$ , where  $d = \gcd(n+m+1, \dots, n+m+1-i, \dots, n+1)$ ,  $a_{m-i} \neq 0$  for some  $i \in \{0, 1, \dots, m\}$ . Thus  $f \equiv tg$  for a constant  $t$  such that  $t^d = 1$ , where  $d = \gcd(n+m+1, \dots, n+m+1-i, \dots, n+1)$ ,  $a_{m-i} \neq 0$  for some  $i \in \{0, 1, \dots, m\}$ . If  $h$  is not constant then  $f$  and  $g$  satisfy the algebraic equation  $R(f, g) \equiv 0$ , where

$$\begin{aligned} R(\omega_1, \omega_2) &= \omega_1^{n+1} \left( \frac{a_m \omega_1^m}{n+m+1} + \frac{a_{m-1} \omega_1^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \\ &\quad - \omega_2^{n+1} \left( \frac{a_m \omega_2^m}{n+m+1} + \frac{a_{m-1} \omega_2^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right). \end{aligned}$$

□

**Proof of Theorem 1.1.** Since  $E_k(0, \psi'(f)) = E_k(0, \psi'(g))$ , it follows that  $E_k(1, F) = E_k(1, G)$ , except the zeros and poles of  $\alpha'$ .

First suppose  $H \neq 0$ . In this case also (23) and (24) hold with  $N_E^1(r, 1; F) = N(r, 1; F | = 1)$ . Using (23), (24) and adopting the same procedure as done in the Proof of Theorem 1 of [19], when  $k = 3$  we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 4\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 2\{N_2(r, 0; F) \\ &\quad + N_2(r, 0; G)\} + S(r, f) + S(r, g). \end{aligned}$$

(28)

Hence by using Lemma 2.2 we get from (28)

$$\begin{aligned}
& (n + m^* + 1)\{T(r, f) + T(r, g)\} \\
\leq & T(r, F) + T(r, G) + N_3(r, 0; F_1) + N_3(r, 0; G_1) - N_2(r, 0; F) - N_2(r, 0; G) \\
& + S(r, f) + S(r, g) \\
\leq & 4\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + N_3(r, 0; f^{n+1}P_1(f)) + N_3(r, 0; g^{n+1}P_1(g)) \\
& + N_2(r, 0; F) + N_2(r, 0; G) + S(r, f) + S(r, g) \\
\leq & 5\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 6\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} + 2m^*\{T(r, f) + T(r, g)\} \\
& + S(r, f) + S(r, g) \\
\leq & (2m^* + 11)T(r) + S(r).
\end{aligned}
\tag{29}$$

In a similar way we can obtain

$$(30) \quad (n + m^* + 1)T(r, g) \leq (2m^* + 11)T(r) + S(r).$$

Combining (29) and (30) we see that

$$(n - m^* - 10) T(r) \leq S(r),$$

which is a contradiction.

When  $k = 2$ , using (23), (24) and proceeding in the same way as in the Proof of Theorem 1 of [19] we get that

$$\begin{aligned}
T(r, F) + T(r, G) & \leq \frac{9}{2}\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 2\{N_2(r, 0; F) + N_2(r, 0; G)\} \\
& + \frac{1}{2}\{\overline{N}(r, 0; F) + \overline{N}(r, 0; G)\} + S(r, f) + S(r, g).
\end{aligned}
\tag{31}$$

(31)

Using Lemma 2.2 we get from (31) that

$$\begin{aligned}
(n + m^* + 1)T(r, f) & \leq 6\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 7\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} \\
& + \frac{5m^*}{2}\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g) \\
& \leq \left(\frac{5m^*}{2} + 13\right) T(r) + S(r).
\end{aligned}
\tag{32}$$

(32)

Similarly, we can obtain

$$(33) \quad (n + m^* + 1) T(r, g) \leq \left(\frac{5m^*}{2} + 13\right) T(r) + S(r).$$

Combining (32) and (33) we see that

$$\left(n - \frac{3m^*}{2} - 12\right) T(r) \leq +S(r),$$



which leads to a contradiction.

Last suppose  $k = 1$ . Using (23), (24) and proceeding in the same way as in the Proof of Theorem 1 of [19] we get that

$$\begin{aligned}
 T(r, F) + T(r, G) &\leq 6\{\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)\} + 2\{N_2(r, 0; F) + N_2(r, 0; G)\} \\
 &\quad + 2\{\overline{N}(r, 0; F) + \overline{N}(r, 0; G)\} + S(r, f) + S(r, g).
 \end{aligned}
 \tag{34}$$

Using Lemma 2.2 we get from (34) that

$$\begin{aligned}
 (n + m^* + 1)T(r, f) &\leq 9\{\overline{N}(r, \infty, f) + \overline{N}(r, \infty; g)\} + 10\{\overline{N}(r, 0; f) + \overline{N}(r, 0; g)\} \\
 &\quad + 4m^*\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g) \\
 &\leq (4m^* + 19) T(r) + S(r).
 \end{aligned}
 \tag{35}$$

In a similar way we can obtain

$$(n + m^* + 1) T(r, g) \leq (4m^* + 19) T(r) + S(r).
 \tag{36}$$

Combining (35) and (36) we get

$$(n - 3m^* - 18) T(r) \leq +S(r),$$

which leads to a contradiction. Next suppose  $H \equiv 0$ . Then by Lemma 2.6 and following the same procedure as adopted in the proof of Theorem 1.2 we can easily deduce the conclusions of the theorem. Hence the proof is completed.  $\square$

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