LINEAR FUNCTIONALS IN SOME SPACES OF ENTIRE FUNCTIONS OF FINITE ORDER

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Abstract. We consider the the linear topology space of entire functions of a proximate order and normal type with respect to the proximate order. We obtain the form of continuous linear functional on this space.

1. Introduction

We introduce the necessary definitions. A function $\rho(r)$, defined on the ray $(0,\infty)$ and satisfying the Lipschitz condition on any segment $[a,b]\subset(0,\infty)$, that satisfies the conditions

$$\lim_{r\to\infty}\rho(r)=\rho\geq 0, \text{ and } \lim_{r\to\infty} r\rho'_+(r)\ln r=0$$

is called *a proximate order*.
A detailed exposition of the properties of proximate order can be found in [**1, 2**]. In this paper we use the notation $V(r)=r^{\rho(r)}$. We will assume that $V(r)$ is an increasing function on $(0,\infty)$ and $\lim\limits_{r\to+0}V(r)=0$.
We now formulate some simple property of proximate order that we shall need frequently [**1**, (2), p.33].
For $r\to\infty$ and $0<a\leq k\leq b<\infty$ asymptotic inequality

$$(1-\varepsilon)k^\rho V(r)<V(kr)<(1+\varepsilon)k^\rho V(r) \tag{1}$$

holds uniformly in k.
Let $M_f(r)=\max\limits_{|z|=r}|f(z)|$. If for the entire function $f(z)$ the quantity

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M_f(r)}{V(r)}$$

is different from zero and infinity, then $\rho(r)$ is called *a proximate order of the given entire function $f(z)$ and σ_f is called the type of the function $f(z)$ with respect to the proximate order $\rho(r)$.*
Let $\rho(r)$ be a proximate order, $\lim_{r\to\infty}\rho(r)=\rho\geq 0$. A single valued function $f(z)$ of the complex variable z is said to belong to the space $[\rho(r),p]$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than or equal p.

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A sequence of functions $\{f_n(z)\}$ from $[\rho(r), p]$ converges in the sense of $[\rho(r), p]$ if (i) it converges uniformly on compacts, (ii) for all $\varepsilon > 0$ there exists $r_0(\varepsilon)$ does not depend on n such that

$$|f_n(z)| < \exp[(p+\varepsilon)V(|z|)], \quad |z| > r_0(\varepsilon)\ (n \geq 1)\,.$$

For a suitable $C(\varepsilon)$, which does not depend on n, for all z

$$|f_n(z)| < C(\varepsilon)\exp[(p+\varepsilon)V(|z|)] \quad (n \geq 1)\,. \tag{2}$$

The space $[\rho(r), p]$ is the linear topology space with sequence topology.
We introduce the function $\varphi(t)$ defined to be the unique solution of the equation $t = V(r)$. So

$$\varphi(V(t)) = t\,. \tag{3}$$

Theorem 1.1. ([**1**, Theorem 2', p.42]) *The type σ_f of the entire function $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ with the proximate order $\rho(r)$ $(\rho > 0)$ is given by the equation*

$$\limsup_{n\to\infty} \varphi(n)\sqrt[n]{|c_n|} = (e\sigma_f\rho)^{1/\rho}\,. \tag{4}$$

Set

$$d_n = \frac{(ep\rho)^{n/\rho}}{(\varphi(n))^n} \quad n \geq 1,\ d_0 = 1\,.$$

For a function $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n \in [\rho(r), p]$, we associate the function

$$F(z) = \sum_{n=0}^{\infty} b_n z^n, \quad b_n = \frac{c_n}{d_n}\ (n \geq 0)\,. \tag{5}$$

It is regular, in any case in the circle $|z| < 1$. Indeed, by (4) we have

$$\limsup_{n\to\infty} \varphi(n)\sqrt[n]{|c_n|} \leq (ep\rho)^{1/\rho},$$

from this $\varphi(n)\sqrt[n]{|c_n|} \leq (ep_1\rho)^{1/\rho}$, $p_1 > p$, $n > n_0$, and

$$|b_n| < \left(\frac{p_1}{p}\right)^{n/\rho}, \quad n \geq n_0\,.$$

Since p_1 is any more p then

$$\limsup_{n\to\infty} \sqrt[n]{|b_n|} \leq 1$$

and the series (5) converges in the circle $|z| < 1$. Conversely, for any analytical function $F(z)$ in the disk $|z| < 1$, the function $f(z)$ from $[\rho(r), p]$ corresponds.
Mapping function $f(z)$ of $[\rho(r), p]$ to the function $F(z)$ as indicated above will be celebrating a record

$$f(z) \sim F(z)\,.$$

It is obvious that if $F(z) \in [\rho(r), p]$ then $f(\lambda z) \sim F(\lambda z)$ in the sense of $[\rho(r), \lambda^\rho p]$, where λ is a parameter, and if $f_n(z) \sim F_n(z)$ $(n = 1, 2, \ldots, m)$ then

$$\sum_{n=1}^{m} a_n f_n(z) \sim \sum_{n=1}^{m} a_n F_n(z)\,.$$

In the present paper we prove two theorems.

Theorem 1.2. *In order to be a sequence* $\{f_n(z)\}$ *of functions from* $[\rho(r), p]$ *to converge in the sense of* $[\rho(r), p]$*, necessary and sufficient condition is that the sequence* $\{F_n(z)\}$ $(f_n(z) \sim F_n(z))$ *converges uniformly inside the disk* $|z| < 1$.

Theorem 1.3. *Continuous linear functional* l *on the space* $[\rho(r), p]$ *has the form*

$$l(f) = \sum_{n=0}^{\infty} a_n c_n, \quad f(z) = \sum_{n=0}^{\infty} c_n z^n\,, \tag{6}$$

where the quantities a_n *satisfy*

$$\limsup_{n\to\infty} \varphi^{-1}(n)\, \sqrt[n]{|a_n|} < (ep\rho)^{-1/\rho}\,. \tag{7}$$

The case $\rho(r) \equiv \rho > 0$ was considered by A.F Leont'ev [**3**, Theorem 1.1.7, Theorem 1.1.9].

2. The space of entire functions $[\rho(r), p]$

We now prove the Theorem 1.2. Let

$$f_k(z) = \sum_{n=0}^{\infty} c_n^{(k)} z^n, \quad F_k(z) = \sum_{n=0}^{\infty} b_n^{(k)} z^n \quad (k \geq 1),$$

and let the sequence $\{f_k(z)\}$ converge in the sense of $[\rho(r), p]$. By Cauchy inequality and the condition (2), we have

$$|c_n^{(k)}| < C(\varepsilon) \frac{\exp[(p+\varepsilon)V(r)]}{r^n}, \quad r > 0\,.$$

Inserting $r = \dfrac{\varphi(n)}{(p_1\rho)^{1/\rho}}$, $p_1 = p + \varepsilon$ into the above inequality, by (1) and (3) we obtain

$$|c_n^{(k)}| < \frac{C(\varepsilon)(p_1\rho)^{n/\rho}}{(\varphi(n))^n} \exp\left[p_1 V\left(\frac{\varphi(n)}{(p_1\rho)^{1/\rho}}\right)\right] \leq \frac{C_1(\varepsilon)(p_1 e^{p_1}\rho)^{n/\rho}}{(\varphi(n))^n} \quad (n \geq 0, k \geq 1)\,.$$

From (2) for $z = 0$

$$|c_0^{(k)}| < C_1(\varepsilon) \quad (k \geq 1)\,.$$

Based on these estimates

$$|b_n^{(k)}| = \frac{|c_n^{(k)}|}{d_n} < C_1(\varepsilon)\left(\frac{p_1 e^{p_1}}{p\, e^p}\right)^{n/\rho} \quad (n \geq 0, k \geq 1)\,.$$

Take $r < \left(\dfrac{p\, e^p}{p_1 e^{p_1}}\right)^{1/\rho}$, $p_1 = p + \varepsilon$. Due to this choice of ε, r may be taken arbitrarily close to unity. We have

$$|F_m(z) - F_k(z)| < \sum_{n=1}^{s} |b_n^{(m)} - b_n^{(k)}| r^n + 2C_1(\varepsilon) \sum_{n=s+1}^{\infty} \left(\frac{p_1 e^{p_1}}{p\, e^p}\right)^{n/\rho} r^n\,.$$

The series standing on the right converges. By ε_1 choose s so that the second term is less than ε_1.

Uniform convergence of $\{f_k(z)\}$ on compacts follows that for each fixed n the coefficient $c_n^{(k)}$ has a limit if $k \to \infty$. Then $b_n^{(k)}$ also has a limit if $k \to \infty$. That is why

$$\sum_{n=1}^{s} |b_n^{(m)} - b_n^{(k)}| r^n < \varepsilon_1$$

if m and k are large. Therefore $|F_m(z) - F_k(z)| < 2\varepsilon_1$, $|z| < r$. □

To prove the second part of the Theorem, let the sequence $\{F_k(z)\}$ converge uniformly in the disk $|z| < 1$. Then for fixed n the coefficient $b_n^{(k)}$ has a limit if $k \to \infty$ and $|F_k(z)| < M$, $|z| < r < 1$. Thus

$$|b_n^{(k)}| < \frac{M}{r^n} \quad (n \geq 0,\ k \geq 1)\,.$$

From this estimation, we have

$$|c_n^{(k)}| = |b_n^{(k)}| d_n < M \frac{p\, e\rho)^{n/\rho}}{(r\varphi(n))^n} \quad (n \geq 0, k \geq 1),$$

and

$$|f_k(z)| \leq M \sum_{n=0}^{\infty} \frac{p\, e\rho)^{n/\rho}}{((r\varphi(n))^n} |z|^n \quad (k \geq 1).$$

Note that the right side does not depend k.

By Theorem 1.1 the function $\displaystyle\sum_{n=0}^{\infty} \frac{(p\, e\rho)^{n/\rho}}{(r\varphi(n))^n} z^n$ is entire function of the order ρ and the type $\dfrac{p}{r^\rho}$. This type is close to p when r is close to 1. Therefore the condition (2) is true.

Since the coefficient $c_n^{(k)}$ has a limit if $k \to \infty$ for each fixed n it is possible to prove similarly that the sequence $\{f_k(z)\}$ converges uniformly on compact sets. So this sequence converges in the sense of $[\rho(r), p]$.

Note that if the sequence $\{f_k(z)\}$ converges to $f(z)$ in the sense of $[\rho(r), p]$ and $\{F_k(z)\}$ converges to $F(z)$ then $f(z) \sim F(z)$.

Remark. If the sequence $\{f_k(z)\}$ converges in the sense of $[\rho(r), p_1]$ then the sequence $\{F_k(z)\}$ converges uniformly in the disk $|z| < \left(\dfrac{p}{p_1}\right)$. The converse is true. One can verify this by tracing the previous calculations.

3. Functional form

By Theorem 1.2 it can be argued that the space $[\rho(r), p]$ of functions $f(z)$ with convergence in the sense of $[\rho(r), p]$ is converted into a space of functions $F(z)$ that are analytic in the disk $|z| < 1$ with the convergence in the sense of uniform convergence on compacts. We use this fact to derive the form of a continuous linear functional on the space $[\rho(r), p]$.

Let $l(f)$ be a continuous linear functional on the space $[\rho(r), p]$. Set $l(z^n) = a_n$ $(n \geq 0)$. Let $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ be a function in $[\rho(r), p]$. Since the series converges in the sense of $[\rho(r), p]$ then, by continuity of the functional,

$$l(f) = \sum_{n=0}^{\infty} c_n l(z^n) = \sum_{n=0}^{\infty} c_n a_n .$$

Hence

$$l(f) = \sum_{n=0}^{\infty} a_n c_n, \quad f(z) = \sum_{n=0}^{\infty} c_n z^n . \tag{8}$$

Now consider the linear functional $L(F)$ on the space of functions $F(z)$ that are analytic in the disk $|z| < 1$ by specifying its equalities:

$$L(z^n) = a_n d_n \quad (n \geq 0); \quad d_n = \frac{(ep\rho)^{n/\rho}}{(\varphi(n))^n} \quad n \geq 1,\ d_0 = 1.$$

We do not yet know that it is a continuous functional.
Let $F(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ be regular in $|z| < 1$. Then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \in [\rho(r), p], \quad c_n = b_n d_n .$$

We have

$$L\left(\sum_{n=0}^{m} b_n z^n\right) = \sum_{n=0}^{m} b_n L(z^n) = \sum_{n=0}^{m} c_n a_n .$$

By (8) the right hand side has a limit if $m \to \infty$. Set

$$L(F) = \sum_{n=0}^{\infty} a_n c_n = \sum_{n=0}^{\infty} a_n d_n b_n . \tag{9}$$

Thus

$$l(f) = L(F), \quad f(z) \sim F(z) . \tag{10}$$

Since $l(f)$ is the continuous linear functional it follows that $L(F)$ is the continuous linear functional.
A continuous linear functional on the space of analytic functions in the unit disk is given by the formula

$$L(F) = \sum_{n=0}^{\infty} p_n b_n, \quad F(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where $\limsup\limits_{n\to\infty} \sqrt[n]{|p_n|} < 1$. Due to this and (9) we obtain $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n| d_n} < 1$ or

$$\limsup_{n\to\infty} \frac{\sqrt[n]{|a_n|}}{\varphi(n)} < (p\, e\rho)^{-1/\rho} . \tag{11}$$

So a continuous linear functional $l(f)$ on the space $[\rho(r), p]$ has the form (8) where values a_n satisfy the condition (11). Converse is also true: if the condition (11) is true then the functional (8) is a continuous linear functional $l(f)$ on the space $[\rho(r), p]$ since by this condition the functional (9) is continuous linear functional.
Theorem 1.3 is proved. □

The space of analytic functions in the unit disk $D = \{z : |z| < 1\}$ with convergence in the sense of uniform convergence on compacts, is denoted by $A(D)$. We note the following known facts:

1) Let $F_n(z) \in A(D)$ $(n \geq 1)$. In order to function $F_0(z) \in A(D)$ to be approximated with arbitrary accuracy by linear combinations of functions $F_n(z)$ (uniformly on compact sets of D), the necessary and sufficient condition is that

$$L(F_n) = 0 \quad (n \geq 1)\,, \tag{12}$$

where $L(F)$ is a continuous linear functional on $A(D)$ and $L(F_0) = 0$. In particular, for the system of the functions $\{F_n(z)\}$ to be complete in $A(D)$, the necessary and sufficient condition is that the equalities (12) should yield $L(F) = 0$ for any function $F(z) \in A(D)$.

2) Let M be a closed set in $A(D)$ which does not coincide with $A(D)$ and the function $F_0(z) \in A(D)$ does not belong to M. Then there exists a functional $L(F)$ with property: $L(F) = 0$ for all $F(z) \in M$ but $L(F_0) \neq 0$.

Note that a closed set M in $A(D)$, by the relation $f(z) \sim F(z)$, corresponds to a closed set in the space $[\rho(r), p]$.
On the basis of the noted facts and also taking into account the equality (10), we obtain the following statements:

1) Let $f_n(z) \in [\rho(r), p]$ $(n \geq 1)$. In order to function $f_0(z) \in [\rho(r), p]$ to be approximated with arbitrary accuracy by linear combinations of functions $f_n(z)$ (in the sense of $[\rho(r), p]$), the necessary and sufficient condition is that

$$l(f_n) = 0 \quad (n \geq 1)\,, \tag{13}$$

where $l(f)$ is a continuous linear functional on $[\rho(r), p]$, and $l(f_0) = 0$. In particular, for the system of the functions $\{f_n(z)\}$ to be complete in $[\rho(r), p]$, the necessary and sufficient is that equalities (13) should yield $l(f) = 0$ for any function $f(z) \in [\rho(r), p]$.

2) Let N be a closed set in $[\rho(r), p]$ which does not coincide with $[\rho(r), p]$ and the function $f_0(z) \in [\rho(r), p]$ does not belong to N. Then there exists a functional $l(f)$ with the property: $l(f) = 0$ for all $(z) \in N$ but $l(f_0) \neq 0$.

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