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EXISTENCE AND UNIQUENESS RESULTS FOR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Ahmed Hamoud[a], Nedal Mohammed[b], Kirtiwant Ghadle[c]

[a] *Department of Mathematics, Taiz University, Taiz-Yemen.*
[b] *Department of Mathematics and Computer Science, Taiz University, Taiz-Yemen.*
[c] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-India.*

Abstract

This paper establishes a study on some important latest innovations in the existence and uniqueness results by means of Banach contraction fixed point theorem for Caputo fractional Volterra-Fredholm integro-differential equations with boundary condition. New conditions on the nonlinear terms are given to pledge the equivalence. Finally, an illustrative example is also presented.

Keywords: Volterra-Fredholm integro-differential equation, Caputo sense, Gronwall-Bellman's inequality, Banach contraction fixed point theorem.
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1. Introduction

In recent years, there has been a growing interest in the linear and nonlinear integro-differential equations which are a combination of differential and integral equations [4, 6, 7, 18, 20, 25]. The nonlinear integro-differential equations play an important role in many branches of nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, electrostatics, biology, chemistry and economics [15] and signal processing [27].

Email addresses: `drahmed985@yahoo.com` (Ahmed Hamoud), `dr.nedal.mohammed@gmail.com` (Nedal Mohammed), `drkpghadle1974@gmail.com` (Kirtiwant Ghadle)

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The challenging work is to find the solution while dealing with Volterra-Fredholm fractional integro-differential equations. Therefore, many researchers have tried their best to use different techniques to find the analytical and numerical solutions of these problems [1, 2, 3, 5, 9, 10, 11, 16, 23, 24, 30].

The study of iterative differential and integro-differential equations is linked to the wide applications of calculus in mathematical sciences. These equations are vital in the study of infection models. Many papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the iterative differential equations and integro-differential equations [17, 18, 21, 22].

Recently, Cheng et al. [8, 22], investigated analytic and exact solutions of an iterative functional differential equation of the type

$$\begin{aligned} &u'(x) = f(x, u(h(x) + g(u(x)))),\\ &u(x_0) = x_0 \end{aligned}$$

Lauran [21], studied the existence and uniqueness results for first order differential and iterative differential equations with deviating argument of the type

$$\begin{aligned} &u'(t) = f(t, u(t), u(u(t)), u(\lambda u(t))),\\ &u(t_0) = x_0 \end{aligned}$$

Kendre et al. [18], studied the existence of solution for iterative integro-differential equations of the type

$$\begin{aligned} &u'(t) = f(t, u(u(t)), \int_{t_0}^{t} k(t,s)u(u(s))ds),\\ &u(t_0) = x_0 \end{aligned}$$

In [17], Ibrahim investigated the existence and uniqueness of solution for iterative differential equations of the type

$$\begin{aligned} &D^{\alpha}u(t) = f(t, u(u(t))),\\ &u(0) = u_0. \end{aligned}$$

Unhale and Kendre [29], established the existence and uniqueness of solution for iterative integro-differential equations of the type

$$\begin{aligned} &D^{\alpha}u(t) = f(t) + \int_{0}^{t} h(t,s)u(\lambda u(s))ds,\\ &u(0) = u_0, \end{aligned}$$

Motivated by these problems, in this paper, we discuss new existence and uniqueness results for nonlinear fractional Volterra-Fredholm integro-differential equation with deviating argument of the type

$$D^{\alpha}u(x) = f(x) + \int_{0}^{x} h(x,s)u(u(s))ds + \int_{0}^{T} k(x,s)u(u(s))ds, \quad x, s \in J := [0,T], \tag{1}$$

with the boundary condition

$$au(0) + bu(T) = c, \quad a, b, c \in \mathbb{R}, \; a + b \neq 0, \tag{2}$$

where $D^{\alpha}(.)$, $0 < \alpha < 1$, is the Caputo fractional derivative, $f(t)$, $h(x,s)$ and $k(x,s)$ are given continuous functions, $u(x)$ is the unknown function to be determined.

The main objective of the present paper is to study the new existence and uniqueness results for iterative nonlinear fractional Volterra-Fredholm integro-differential equation with deviating argument.

The rest of the paper is organized as follows: In Section 2, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Section 3, the new existence and uniqueness results of the solution for nonlinear fractional Volterra-Fredholm integro-differential equation have been proved. In Section 5, focuses on an example to illustrate the theory. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

In this segment, we first survey some fundamental definitions of the fractional calculus theory which are required for building up our outcomes. For more details, see [9, 11, 12, 13, 19, 20, 26, 28, 30].

Definition 2.1. *[18] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. *[18] The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$$^{L}D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt, \; x>0, \; 0<\alpha<1. \tag{4}$$

Definition 2.3. *[14] The Caputo derivative of order α for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(t)}{(x-t)^{\alpha}} dt, \; x>0, \; 0<\alpha<1.$$

Definition 2.4. *[14] The fractional derivative of $f(x)$ in the Caputo sense is defined by*

$$\begin{aligned} ^{c}D^{\alpha} f(x) &= J^{n-\alpha} D^n f(x) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, & n-1<\alpha<n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n, \end{cases} \end{aligned} \tag{5}$$

where the parameter α is the order of the derivative, in general it is real or even complex.

Definition 2.5. *[14] The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha} f(x) = D^m J^{m-\alpha} f(x), \qquad m-1<\alpha \leq m. \tag{6}$$

Lemma 2.1. *[14] (Gronwall-Bellman's Inequality). Let $u(x)$ and $f(x)$ be nonnegative continuous functions defined on $J = [\alpha, \alpha+h]$ and c be a nonnegative constant. If*

$$u(x) \leq c + \int_{\alpha}^{x} f(s) u(s) ds, \; x \in J,$$

then

$$u(x) \leq c \, \exp\Big(\int_{\alpha}^{x} f(s) ds \Big), \; x \in J,$$

Theorem 2.1. *[28] (Banach contraction principle). Let (X, d) be a complete metric space, then each contraction mapping $\mathcal{T} : X \longrightarrow X$ has a unique fixed point x of $\mathcal{T}$ in X i.e. $\mathcal{T}x = x$.*

3. Main Results

In this section, we shall give the existence and uniqueness results of Eq.(1), with the condition (2). Let $B = C(J, J)$ be the Banach space equipped with the norm $\|u\| = \max_{x\in[0,T]} |u(x)|$. For convenience, we are listing the following hypotheses used in our further discussion:

(A1) There exists constants β_h and β_k such that

$$\beta_h = \sup\{|h(t,s)| : 0 \le s \le t \le T\}.$$

$$\beta_k = \sup\{|k(t,s)| : 0 \le s \le t \le T\}.$$

(A2) There exists a constant $M > 0$ such that

$$|u(t_1) - u(t_2)| \le M|t_1 - t_2|^{\alpha}, \text{ for } u \in B, \; t_1, t_2 \in J, \; t_1 \le t_2.$$

(A3) There exists a constant $L > 0$ such that $L = \sup\{|f(t)| : 0 \le t \le T\}$.

(A4) Let $\rho := \frac{T^{\alpha}(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)}\Big[1 + \frac{|b|}{|a+b|}\Big] + \frac{|c|}{|a+b|} \le T \le M$.

Lemma 3.1. *If a function $u \in C[0,T]$ satisfies (1)-(2) in the closed interval $[0,T]$, then the problems (1)-(2) are equivalent to the problem of finding a continuous solution of the integral equation*

$$\begin{aligned} u(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt \\ &\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - c\Big]. \end{aligned}$$

Theorem 3.1. *Suppose that the hypotheses (A1)–(A4) are satisfied and*

$$\Big[\frac{T^{\alpha+1}(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\Big(1 + \frac{|b|}{|a+b|}\Big)\Big] < 1.$$

Then there is a unique solution to the problems (1)-(2).

Proof. Let $B(\rho) = \{u \in B : \; 0 \le u \le \rho, |u(t_1) - u(t_2)| \le M|t_1 - t_2|^{\alpha}\}$.

To apply Banach contraction principle, we define an operator $\Psi : B(\rho) \longrightarrow B(\rho)$ by

$$\begin{aligned} (\Psi u)(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt \\ &\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - c\Big]. \end{aligned}$$

So, we have

$$\begin{aligned}
0 \leq |\Psi u| &= \Big| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u(u(s)) ds + \int_0^T k(t,s) u(u(s)) ds \Big) dt \\
&\quad - \frac{1}{a+b} \Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u(u(s)) ds + \int_0^T k(t,s) u(u(s)) ds \Big) dt - c \Big] \Big| \\
&\leq \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(|f(t)| + \int_0^t |h(t,s)| |u(u(s))| ds + \int_0^T |k(t,s)| |u(u(s))| ds \Big) dt \\
&\quad + \frac{1}{|a+b|} \int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(|f(t)| + \int_0^t |h(t,s)| |u(u(s))| ds \\
&\quad + \int_0^T |k(t,s)| |u(u(s))| ds \Big) dt + \frac{|c|}{|a+b|} \\
&\leq \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} (L + (\beta_h + \beta_k) T^3) ds + \frac{1}{|a+b|} \int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)} (L + (\beta_h + \beta_k) T^3) dt \\
&\quad + \frac{|c|}{|a+b|} \\
&\leq \frac{T^\alpha (L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} \Big[1 + \frac{|b|}{|a+b|} \Big] + \frac{|c|}{|a+b|} \\
&= \rho.
\end{aligned}$$

Also, for each $0 \leq x_1 \leq x_2 \leq T$, we have

$$\begin{aligned}
&\Big| \Psi u(x_2) - \Psi u(x_1) \Big| \\
\leq\ & \Big| \int_0^{x_1} \frac{(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u(u(s)) ds + \int_0^T k(t,s) u(u(s)) ds \Big) dt \Big| \\
+\ & \Big| \int_{x_1}^{x_2} \frac{(x_2-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u(u(s)) ds + \int_0^T k(t,s) u(u(s)) ds \Big) dt \Big| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \Big] \Big(\Big| f(t) \Big| + \int_0^t \Big| h(t,s) \Big| \Big| u(u(s)) \Big| ds \\
+\ & \int_0^T \Big| k(t,s) \Big| \Big| u(u(s)) \Big| ds \Big) dt + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \Big(\Big| f(t) \Big| + \int_0^t \Big| h(t,s) \Big| \Big| u(u(s)) \Big| ds \\
+\ & \int_0^T \Big| k(t,s) \Big| \Big| u(u(s)) \Big| ds \Big) dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big| \Psi u(x_2) - \Psi u(x_1) \Big| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \Big] [L + T^3(\beta_h + \beta_k)] dt \\
+\ & \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} [L + T^3(\beta_h + \beta_k)] dt \\
\leq\ & \frac{[L + T^3(\beta_h + \beta_k)]}{\Gamma(\alpha+1)} \Big[x_1^\alpha - x_2^\alpha + 2(x_2-x_1)^\alpha \Big] \\
\leq\ & \frac{2[L + T^3(\beta_h + \beta_k)]}{\Gamma(\alpha+1)} \Big| x_2 - x_1 \Big|^\alpha
\end{aligned}$$

This shows that Ψ maps from $B(\rho) \longrightarrow B(\rho)$, Now, for all $u, v \in B(\rho)$ we have

$$
\begin{aligned}
& \Big|\Psi u(x) - \Psi v(x)\Big| \\
\leq \ & \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t \Big|h(t,s)\Big|\Big|u(u(s)) - v(v(s))\Big| ds \\
+ \ & \int_0^T \Big|k(t,s)\Big|\Big|u(u(s)) - v(v(s))\Big| ds \Big) dt \\
+ \ & \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t \Big|h(t,s)\Big|\Big|u(u(s)) - v(v(s))\Big| ds \\
+ \ & \int_0^T \Big|k(t,s)\Big|\Big|u(u(s)) - v(v(s))\Big| ds \Big) dt \\
\leq \ & \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t \Big|u(u(s)) - u(v(s))\Big| + \Big|u(v(s)) - v(v(s))\Big| ds \\
+ \ & \int_0^T \Big|u(u(s)) - u(v(s))\Big| + \Big|u(v(s)) - v(v(s))\Big| ds \Big) dt \\
+ \ & \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t \Big|u(u(s)) - u(v(s))\Big| + \Big|u(v(s)) - v(v(s))\Big| ds \\
+ \ & \int_0^T \Big|u(u(s)) - u(v(s))\Big| + \Big|u(v(s)) - v(v(s))\Big| ds \Big) dt \\
\leq \ & \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t (M|u(s) - v(s)| + |u(s) - v(s)|) ds \Big) dt \\
+ \ & \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t (M|u(s) - v(s)| + |u(s) - v(s)|) ds \Big) dt \\
\leq \ & \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t ((M+1)|u(s) - v(s)|) ds \Big) dt \\
+ \ & \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t ((M+1)|u(s) - v(s)|) ds \Big) dt \\
\leq \ & \frac{T(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha)} \|u - v\| \int_0^x (x-t)^{\alpha-1} dt + \frac{|b| T(\beta_h + \beta_k)(M+1)}{|a+b|\Gamma(\alpha)} \|u - v\| \int_0^T (T-t)^{\alpha-1} dt \\
\leq \ & \frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \|u - v\| + \frac{|b| T T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{|a+b|\Gamma(\alpha+1)} \|u - v\| \\
\leq \ & \Big[\frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \Big(1 + \frac{|b|}{|a+b|}\Big) \Big] \|u - v\|
\end{aligned}
$$

Since

$$\Big[\frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \Big(1 + \frac{|b|}{|a+b|}\Big) \Big] < 1,$$

by the Banach contraction principle, Ψ has a unique fixed point. This means that the problems (1)-(2) has unique solution.

□

The above theorem shows that there exists a unique solution to the problems (1)-(2). However, it does not tell us how to find this solution. To find the solution of the problems (1)-(2), we will define the following sequence

$$\begin{aligned} & u_{n+1}(x) \\ = \; & \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_n(u_n(s))ds+\int_0^T k(t,s)u_n(u_n(s))ds\Big)dt \\ & -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_n(u_n(s))ds+\int_0^T k(t,s)u_n(u_n(s))ds\Big)dt-c\Big]. \end{aligned} \tag{7}$$

where $n = 0, 1, 2, ...$ and $u_0(x)$ is fixed functions of the class C^1 mapping $[0, T] \longrightarrow [0, T]$ such that $|u_0(x)| \leq T$. For this, we have the following theorem.

Theorem 3.2. *If the assumptions of the Theorem 3.1 are satisfied then the sequences defined in* (7) *converges uniformly to the unique solution of the problems* (1)-(2).

Proof. Let $U_k = \max_{x\in J} |u_k(x) - u_{k-1}(x)|$. Then

$$\begin{aligned} U_1 &= \max_{x\in J}|u_1(x)-u_0(x)| \\ &= \max_{x\in J}\Big|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_0(u_0(s))ds+\int_0^T k(t,s)u_0(u_0(s))ds\Big)dt-\frac{1}{a+b} \\ &\times \Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_0(u_0(s))ds+\int_0^T k(t,s)u_0(u_0(s))ds\Big)dt-c\Big]-u_0(x)\Big| \\ &\leq \frac{T^\alpha(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)}\Big(1+\frac{|b|}{|a+b|}\Big)+\frac{|c|}{|a+b|} \\ &\leq T. \end{aligned}$$

Since $u_0 : [0, T] \longrightarrow [0, T]$, we have $U_1 \leq T$.

$$\begin{aligned} U_2 &= \max_{x\in J}|u_2(x)-u_1(x)| \\ &= \max_{x\in J}\Big|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_1(u_1(s))ds+\int_0^T k(t,s)u_1(u_1(s))ds\Big)dt \\ &\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_1(u_1(s))ds+\int_0^T k(t,s)u_1(u_1(s))ds\Big)dt-c\Big] \\ &\quad -\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_0(u_0(s))ds+\int_0^T k(t,s)u_0(u_0(s))ds\Big)dt \\ &\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_0(u_0(s))ds+\int_0^T k(t,s)u_0(u_0(s))ds\Big)dt-c\Big]dt\}\Big| \\ &\leq \max_{x\in J}\Big\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big|f(t)\Big|+\int_0^t \Big|h(t,s)\Big|\Big|u_1(u_1(s))-u_0(u_0(s))\Big|ds \\ &\quad +\int_0^T \Big|k(t,s)\Big|\Big|u_1(u_1(s))-u_0(u_0(s))\Big|ds\Big)dt-\frac{1}{\Big|a+b\Big|}\int_0^T \frac{\Big|b\Big|(T-t)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad \times\Big(\Big|f(t)\Big|+\int_0^t \Big|h(t,s)\Big|\Big|u_1(u_1(s))-u_0(u_0(s))\Big|ds+\int_0^T \Big|k(t,s)\Big|\Big|u_1(u_1(s))-u_0(u_0(s))\Big|ds\Big)dt\Big\} \\ &\leq TU_1 \leq T^2. \end{aligned}$$

Assume that result is true for n i.e. $U_n \leq TU_{n-1} \leq T^n$. Now, we show that result holds for $n + 1$

$$
\begin{aligned}
U_{n+1} &= \max_{x\in J}|u_{n+1}(x)-u_n(x)| \\
&= \max_{x\in J}\Big|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_1(u_n(s))ds+\int_0^T k(t,s)u_n(u_n(s))ds\Big)dt \\
&\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_n(u_n(s))ds+\int_0^T k(t,s)u_n(u_n(s))ds\Big)dt-c\Big] \\
&\quad -\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_{n-1}(u_{n-1}(s))ds+\int_0^T k(t,s)u_{n-1}(u_{n-1}(s))ds\Big)dt \\
&\quad -\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_{n-1}(u_{n-1}(s))ds \\
&\quad +\int_0^T k(t,s)u_{n-1}(u_{n-1}(s))ds\Big)dt-c\Big]dt\}\Big| \\
&\le \max_{x\in J}\Big\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big|f(t)\Big|+\int_0^t \Big|h(t,s)\Big|\Big|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))\Big|ds \\
&\quad +\int_0^T \Big|k(t,s)\Big|\Big|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))\Big|ds\Big)dt \\
&\quad -\frac{1}{\Big|a+b\Big|}\int_0^T \frac{\Big|b\Big|(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big|f(t)\Big|+\int_0^t \Big|h(t,s)\Big|\Big|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))\Big|ds \\
&\quad +\int_0^T \Big|k(t,s)\Big|\Big|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))\Big|ds\Big)dt\Big\} \\
&\le TU_n \le T^{n+1}.
\end{aligned}
$$

Thus by induction, we have $U_k \le T^k$. Since

$$u_0+\frac{T^\alpha(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)} \le T < 1, \text{ when } u_0 \ge 0.$$

Hence U_k tends to zero as k tends to infinity. Since the family $\{U_k\}$ is the Arzelà-Ascoli family thus for every subsequence $\{u_{kj}\}$ of $\{U_k\}$ there exists a subsequence $\{u_{kj}\}$ uniformly convergent and the limit needs to be a solution of the problem (1)-(2). Thus, the sequence $\{U_k\}$ tends uniformly to the unique solution of the problem (1)-(2). □

4. Ulam-Hyers Stability

In this section, we investigate the Ulam-Hyers stability and generalized Ulam-Hyers stability for the problem (1)-(2).

Definition 4.1. *The Eq.* (1) *is Ulam-Hyers stable if there exists a real number* $\Theta > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $v \in C^1(J,J)$ *of the inequality*

$$\Big|D^\alpha v(x)-f(x)-\int_0^x h(x,s)v(v(s))ds-\int_0^T k(x,s)v(v(s))ds\Big| \le \epsilon, \;\; x \in J, \tag{8}$$

there exists a solution $u \in C^1(J,J)$ *of Eq.* (1) *with*

$$|v(x)-u(x)| \le \Omega\epsilon. \tag{9}$$

Definition 4.2. *The Eq.* (1) *is generalized Ulam-Hyers stable if there exists* $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\Theta(0) = 0$ *such that for each* $\epsilon > 0$ *and for each solution* $v \in C^1(J, J)$ *of the inequality*

$$\left| D^\alpha v(x) - f(x) - \int_0^x h(x,s)v(v(s))ds - \int_0^T k(x,s)v(v(s))ds \right| \leq \epsilon, \;\; x \in J, \tag{10}$$

there exists a solution $u \in C^1(J, J)$ *of Eq.* (1) *with*

$$|v(x)u(x)| \leq \Theta(\epsilon). \tag{11}$$

Theorem 4.1. *If the assumptions of the Theorem 3.1 are satisfied, then the problem* (1)-(2) *is Ulam-Hyers stable.*

Proof. let $\epsilon > 0$ and let function $v \in C^1(J, J)$ which satisfies the inequality

$$D^\alpha v(x) - f(x) - \int_0^x h(x,s)v(v(s))ds - \int_0^T k(x,s)v(v(s))ds \leq \epsilon, \tag{12}$$

and let $u \in C(J, J)$ be the unique solution of the following problem

$$\begin{aligned} D^\alpha u(x) &= f(x) + \int_0^x h(x,s)u(u(s))ds + \int_0^T k(x,s)u(u(s))ds, \\ & u(0) = v(0), \quad u(T) = V(T). \end{aligned}$$

from Lemma 3.1, we obtain

$$\begin{aligned} u(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt \\ &\quad - \frac{1}{a+b} \Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt - c \Big] \\ &= \Delta_u + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt, \end{aligned}$$

where

$$\Delta_u = \frac{1}{a+b} \Big[c - \int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt \Big].$$

Let

$$\Delta_v = \frac{1}{a+b} \Big[c - \int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds \Big) dt \Big].$$

On the other hand, if $u(0) = v(0)$, $u(T) = v(T)$, then $\Delta_u = \Delta_v$ and

$$u(x) = \Delta_v + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt.$$

From inequality (12) we have

$$-\epsilon \leq D^\alpha v(x) - f(x) - \int_0^x h(x,s)v(v(s))ds - \int_0^T k(x,s)v(v(s))ds \leq \epsilon, \tag{13}$$

If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$\left| v(x) - \Delta_v - \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds \Big) dt \right| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}.$$

For any $x \in J$, we have

$$\begin{aligned}
&\Big|v(x) - u(x)\Big| \\
\leq\ & \Big|v(x) - \Delta_v - \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds\Big)dt\Big| \\
&+ \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^t h(t,s)|v(v(s)) - u(u(s))|ds + \int_0^T k(t,s)|v(v(s)) - u(u(s))|ds\Big)dt \\
\leq\ & \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\beta_h \int_0^t [|v(v(s)) - v(u(s))| + |v(u(s)) - u(u(s))|]ds \\
&+\beta_k \int_0^T [|v(v(s)) - v(u(s))| + |v(u(s)) - u(u(s))|]ds\Big)dt \\
\leq\ & \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\beta_h}{\Gamma(\alpha)}\int_0^x\int_0^t (x-t)^{\alpha-1}(M+1)|v(s) - u(s)|dsdt \\
&+\frac{\beta_k}{\Gamma(\alpha)}\int_0^x\int_0^T (x-t)^{\alpha-1}(M+1)|v(s) - u(s)|dsdt \\
\leq\ & \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{\alpha}(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\int_0^x |v(s) - u(s)|ds.
\end{aligned}$$

Using Gronwall's inequality, we get

$$\Big|v(x) - u(x)\Big| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}\Big[1 + \frac{\xi T^{\alpha}(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\Big] : \Omega\epsilon,$$

where $\xi = \xi(\alpha)$ a constant, which completes the proof.

Moreover, if we set $\Theta(\epsilon) = \Omega\epsilon$, $\Theta(0) = 0$, then boundary value problem (1)-(2) is generalized Ulam-Hyers stable. □

5. An Example

We consider the nonlinear iterative fractional integro-differential equation (1)-(2) with

$$\alpha = 0.5,\ T = 0.5, L = 0.2, M = 0.4, \beta_h = \beta_k = 0.5,\ a = b = 1, \text{and } c = 0.$$

New, we have

$$\begin{aligned}
\frac{T^{\alpha}(L + T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)}\Big(1 + \frac{|b|}{|a+b|}\Big) + \frac{|c|}{|a+b|} &= \frac{0.5^{0.5}(0.2 + 0.5^3(0.5+0.5))}{\Gamma(0.5+1)}(1 + \frac{1}{2}) + 0 \\
&= \frac{0.2298098}{\Gamma(1.5)}(1.5) \\
&= \frac{0.3447145}{0.886227} \\
&= 0.38897 \\
&< 0.5 = T.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{T^{\alpha+1}(M+1)(\beta_h+\beta_k)}{\Gamma(\alpha+1)}\Big(1 + \frac{|b|}{|a+b|}\Big) &= \frac{0.5^{0.5+1}(0.4+1)(0.5+0.5)}{\Gamma(0.5+1)}(1 + \frac{1}{2}) \\
&= \frac{0.494975}{0.886227}(1.5) \\
&= 0.8378 \\
&< 1.
\end{aligned}$$

Since all the hypotheses of Theorem 3.1 are fulfilled, then there exists a unique solution of the given equation.

6. Conclusion

The main purpose of this paper was to present new existence and uniqueness results as well as the Ulam-Hyers stability and generalized Ulam-Hyers stability results of the solution for Caputo fractional iterative Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as the Gronwall-Bellman's inequality, some properties of fractional calculus and the Banach contraction fixed point theorem. Moreover, the results of references [17, 18, 29] appear as a special case of our results.

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