Abstract. In this paper, we introduce the grand Lorentz sequence spaces \( \ell^{p,q}_G \) and study on some topological properties. Also, we characterize some properties of the multiplication operator, such as compactness, Fredholmness etc., defined on \( \ell^{p,q}_G \).

1. Introduction

Let \((X, S, \mu)\) be a \(\sigma\)-finite measure space and let \(g\) be a complex-valued measurable function defined on \(X\). The non-increasing rearrangement \(g^*\) of \(g\) is defined by

\[
g^*(s) = \inf \{ t > 0 : F_\mu(t) \leq s \}, \quad s \geq 0,
\]

where \(F_\mu(t) = \mu \{ x \in X : |g(x)| > t \}, t \geq 0, \) is the distribution function of \(g\). If \(\mu\) is counting measure on \(S = 2^N\), then we can write the distribution function and the non-increasing rearrangement of a complex-valued sequence \((x_n)\), respectively, as follows;

\[
F_\mu(t) = \mu \{ n \in \mathbb{N} : |x_n| > t \}, \quad t \geq 0
\]

and

\[
x_{\phi(n)} = \inf \{ t > 0 : F_\mu(t) \leq n - 1 \}
\]

if \(n-1 \leq t < n\) with \(F_\mu(t) < \infty\). By the definition of non-increasing rearrangement, we can interpret that \((x_{\phi(n)})\) can be obtained by permuting \((|x_n|)_{n \in \mathbb{N}}\), where \(R = \{ n \in \mathbb{N} : x_n \neq 0 \}\), in the decreasing order. Here, \(x_{\phi(n)} = 0\) for \(n > \mu(R)\) if \(\mu(R) < \infty\).
Lorentz introduced the classical Lorentz space $\Lambda_{q,w}$, $0 < q < \infty$, which is the space of all measurable functions $f$ defined on $(0,1)$ with
\[
\|f\|_{\Lambda_{q,w}} = \left( \int_0^1 (f^*(x))^q w(x) \, dx \right)^{\frac{1}{q}},
\]
where $f^*$ is the non-increasing rearrangement of $f$ and $w$ is a weight function \[12\], \[13\]. The space $\Lambda_{q,w}$ and its special case $L^{p,q}$, $0 < q,p \leq \infty$, have been widely studied by many authors. For more details see \[3\], \[5\], \[7\].

The Lorentz sequence spaces $\ell_{p,q}$ is the space of all complex-valued sequences $x = (x_n)$ such that
\[
\|x\|_{\ell_{p,q}} = \left\{ \left( \sum_{n=1}^{\infty} n^{\frac{p}{q}-1} (x_{\phi(n)})^q \right)^{\frac{1}{q}}, \quad 1 \leq p \leq \infty, \ 1 \leq q < \infty \right\} \sup_n n^{\frac{1}{q}} x_{\phi(n)}, \quad 1 \leq p < \infty, \ q = \infty
\]
is finite, where $(x_{\phi(n)})$ is non-increasing rearrangement of $x$. The spaces $\ell_{p,q}$ have been used to introduce and investigate some classes of operators, like $(p,q)$--nuclear, $(p,q;r)$--absolutely summing operator \[14\]. Kato \[11\] characterized the dual space of $\ell_{p,q}(E)$, where $E$ is a Banach space. See also \[2\], \[10\], \[15\].

The idea of grand spaces was raised by Iwaniec and Sbordone \[8\]. They introduced the grand Lebesgue spaces $L^{(p)}$ for $1 < p < \infty$. Samko and Umakhatzhev \[17\] studied some properties of grand Lebesgue spaces on sets of infinite measure. Jain and Kumari \[9\] introduced the grand Lorentz spaces $\Lambda_{q,w}^G$, $0 < q < \infty$ and studied on its basic properties. Also, they characterized boundedness of maximal operator on the space $\Lambda_{q,w}$. Later, Rafeiro and others \[16\] introduced the grand Lorentz sequence space $\ell^{(p)}(X)$ by the norm
\[
\|x\|_{\ell^{(p)}(X)} = \sup_{\varepsilon > 0} \left( \varepsilon^{p} \sum_{k \in X} |x_k|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}, \quad X \text{ is one of the sets } \mathbb{Z}^n, \mathbb{Z}, \mathbb{N} \text{ and } \mathbb{N}_0 \text{ for } 1 \leq p < \infty, \theta > 0. \text{ They studied various operators of harmonic analysis, e. g. maximal, convolution, Hardy etc.}
\]

In this paper, we are inspired by this work and introduce the grand Lorentz sequence spaces $\ell^{(p)}_{p,q}$ as follows; let $\theta > 0$. The grand Lorentz sequence space $\ell^{(p)}_{p,q}$ is the set of all sequences $a = (a_n)$ such that $\|a\|_{(p,q),\theta} < \infty$, where $\|a\|_{(p,q),\theta}$ is defined by
\[
\sup_{\varepsilon > 0} \left( \varepsilon^{p} \sum_{n=1}^{\infty} \left( \frac{1}{n^{p(1+\varepsilon)}} a_{\phi(n)} \right)^q (1+\varepsilon) n^{-1} \right)^{\frac{1}{q(1+\varepsilon)}} = \sup_{\varepsilon > 0} \varepsilon^{p} \sum_{n=1}^{\infty} n^{\frac{1}{q(1+\varepsilon)}} a_{\phi(n)}, \quad \sup_{n \geq 1} n^{\frac{1}{q(1+\varepsilon)}} a_{\phi(n)}, \quad 1 \leq p < \infty, q = \infty
\]
where $(a_{\phi(n)})$ is the non-increasing rearrangement of the sequence $a = (a_n)$. In case $p = q$, the grand Lorentz sequence space $\ell^{(p)}_{p,q}$ coincides with the grand Lebesgue.
space $\ell^p(\mathbb{N})$. In this work, we study on some topological properties and inclusion theorems of the space $\ell^\theta_{p,q}$). Also, we characterize some properties of multiplication operator on the $\ell^\theta_{p,q}$.

We will need the following lemma:

**Lemma 1.** (Hardy, Littlewood and Polya) Let $(r_n^*)$ and $(^*r_n)$ be the non-increasing and non-decreasing rearrangements of a finite sequence $(r_n)$ of positive numbers. Then, we have for any two sequences $(a_n)$ and $(b_n)$ of positive numbers such that

$$
\sum_n a_n^* b_n \leq \sum_n a_n b_n \leq \sum_n a_n^* b_n^* 
$$

[6].

2. Main Results

2.1. Grand Lorentz Sequence Space.

**Theorem 2.** The grand Lorentz sequence space $\ell^\theta_{p,q}$ is a normed space for $1 \leq q \leq p \leq \infty$ and a quasi-normed space for $1 \leq p < q \leq \infty$.

**Proof.** By definition of the norm of $\ell^\theta_{p,q}$, we can write

$$
\|a\|_{\ell^\theta_{p,q}} = \sup_{\varepsilon > 0} \varepsilon^{\frac{1}{\theta(1+\varepsilon)}} \|a\|_{\ell_{p,q}(1+\varepsilon)}. 
$$

Let $1 \leq q < p \leq \infty$. For any $a, b \in \ell^\theta_{p,q}$, since $n^{\frac{1}{q}-1}$ is decreasing sequence of positive numbers and so by Lemma 1, we have

$$
\|a + b\|_{\ell^\theta_{p,q}} = \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^\infty n^{\frac{1}{q}-1} (a_{\phi(n)} + b_{\phi(n)})^{q(1+\varepsilon)} \right)^{\frac{1}{\theta(1+\varepsilon)}} 
$$

$$
\leq \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^\infty n^{\frac{1}{q}-1} (a_{\phi(n)})^{q(1+\varepsilon)} \right)^{\frac{1}{\theta(1+\varepsilon)}} + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^\infty n^{\frac{1}{q}-1} (b_{\phi(n)})^{q(1+\varepsilon)} \right)^{\frac{1}{\theta(1+\varepsilon)}} 
$$

$$
\leq \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^\infty n^{\frac{1}{q}-1} (a_{\phi(n)})^{q(1+\varepsilon)} \right)^{\frac{1}{\theta(1+\varepsilon)}} + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^\infty n^{\frac{1}{q}-1} (b_{\phi(n)})^{q(1+\varepsilon)} \right)^{\frac{1}{\theta(1+\varepsilon)}} 
$$

[6].
\[ = \|a\|_{p,q,\theta} + \|b\|_{p,q,\theta} \]

where \((a_{\phi(n)} + b_{\phi(n)})\), \((a_{\psi(n)})\) and \((b_{\psi(n)})\) are the non-increasing rearrangements of \((a_n + b_n)\), \((a_n)\) and \((b_n)\), respectively.

Let \(1 \leq p < q < \infty\). Then, we have \(p < q(1 + \varepsilon)\) for \(\varepsilon > 0\) and hence \(\|a\|_{p,q(1+\varepsilon)}\) is a quasi-norm. Thus, we get

\[ \|a + b\|_{p,q,\theta} = \sup_{\varepsilon > 0} \varepsilon^{\frac{\theta}{\theta+1}} \|a + b\|_{p,q(1+\varepsilon)} \]

\[ \leq \sup_{\varepsilon > 0} \varepsilon^{\frac{\theta}{\theta+1}} \left( 2^\frac{\theta}{\theta+1} \left( \|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)} \right) \right) \]

\[ \leq 2^\frac{\theta}{\theta+1} \left( \|a\|_{p,q,\theta} + \|b\|_{p,q,\theta} \right). \]

For \(1 \leq p < q\) and \(q = \infty\), we have \(\|a\|_{p,\infty,\theta} = \|a\|_{p,\infty}\). The proof is completed. \(\square\)

**Remark 3.** Let \(\alpha > 0\) and let us take the sequence

\[ (a_n) = \left( n^\frac{1}{\alpha} \left( \ln(n + 1) \right)^{-\alpha} \right) \]

as in \([10]\). It is easy to see that the sequence \((a_n)\) is decreasing and thus the non-increasing rearrangement of \((a_n)\) is itself. Therefore, we have

\[ \sum_{n=1}^{\infty} \left( n^\frac{1}{\alpha} n^{-\alpha} \left( \ln(n + 1) \right)^{-\alpha} \right)^q n^{-1} = \sum_{n=1}^{\infty} n^{-1} \left( \ln(n + 1) \right)^{-\alpha q}. \]

If \(\alpha > \frac{1}{q}\), then \((a_n) \in \ell_{p,q}\). Using similar technique as in \([10]\), we get \((a_n) \in \ell^\theta_{p,q}\) if and only if \(\alpha \geq \frac{1-\theta}{q}\). Thus, we get \((a_n) \in \ell^\theta_{p,q}\) and \((a_n) \notin \ell_{p,q}\) whenever \(\frac{1-\theta}{q} \leq \alpha \leq \frac{1}{q}\).

**Definition 4.** The vanishing grand Lorentz sequence space \(\ell^\theta_{p,q}\), \(1 \leq p \leq \infty\), \(1 \leq q < \infty\), consists of all sequences \((a_n) \in \ell^\theta_{p,q}\) such that

\[ \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \left( n^{\frac{1}{\alpha+\varepsilon}} a_{\phi(n)} \right)^{\theta(q+1)} n^{-1} = 0. \]

**Lemma 5.** The space \(\ell^\theta_{p,q}\) is a closed subspace of the space \(\ell^\theta_{p,q}\).

**Proof.** The proof can be obtained by using similar technique as in \([10]\). \(\square\)

**Remark 6.** It is enough to take the supremum in (1) on the finite interval for \(\varepsilon\), which means

\[ \|a\|_{p,q,\theta} = \sup_{0 < \varepsilon < W(\frac{1}{\theta})} \varepsilon^{\frac{\theta}{\theta+1}} \|a\|_{p,q(1+\varepsilon)} \]

where \(W(t)\) is the Lambert function. Note that \(\frac{1}{W(1/\theta)} \approx 3.59\) (see \([4], [10]\)).
Lemma 7. Let $a = (a_n) \in \ell^{p,q}_{p,q}$, $1 \leq p,q < \infty$ and $\theta > 0$. Then, we have the following inequalities for all $n \in \mathbb{N}$:

$$a_{\phi(n)} \leq h \left( \frac{1}{W(e^{-1})} \right)^{-\frac{\theta}{q}} \left( p \right)^{-\frac{1}{q}} R(\varepsilon_0)^{-\frac{1}{q}} n^{-\frac{1}{p}} \|a\|_{p,q,\theta}$$

if $1 \leq p \leq q < \infty$ and

$$a_{\phi(n)} \leq h \left( \frac{1}{W(e^{-1})} \right)^{-\frac{\theta}{q}} n^{-\frac{1}{\theta}} \|a\|_{p,q,\theta}$$

if $1 \leq q < p \leq \infty$, where $h(x) = x^{\frac{1}{1-x}}$, $R(x) = (1 + x)^{-\frac{1}{x}}$ and $\varepsilon_0 \approx 1,7182$.

Proof. Let $a = (a_n) \in \ell^{p,q}_{p,q}$ and let $1 \leq p \leq q < \infty$. Since $p \leq q(1 + \varepsilon)$, we have by Lemma 2 in [11] that

$$\|a\|_{p,q,\theta} = \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)}$$

$$\geq \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left( n^{\frac{1}{p}} \left( p \right)^{\frac{1}{q}} \left( 1 + \varepsilon \right)^{-\frac{1}{q(1+\varepsilon)}} a_{\phi(n)} \right)$$

$$\geq \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left( p \right)^{\frac{1}{q}} (R(\varepsilon))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}$$

$$= \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left( p \right)^{\frac{1}{q}} (R(\varepsilon))^{\frac{1}{q}} n^{\frac{1}{p}} a_{\phi(n)}$$

Here $R(x) = (1 + x)^{-\frac{1}{x}}$ attains the minimum at the point $\varepsilon_0 \approx 1,7182$.

Let $1 \leq q < p < \infty$. Then, since $n^{\frac{2}{p} - 1}$ is decreasing, we have

$$\|a\|_{p,q,\theta} = \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \|a\|_{p,q(1+\varepsilon)}$$

$$\geq \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left( \sum_{n=1}^{k} \left( n^{\frac{1}{p(1+\varepsilon)}} a_{\phi(n)} \right)^{q(1+\varepsilon)} n^{-1} \right)^{\frac{1}{q(1+\varepsilon)}}$$

$$\geq a_{\phi(k)} \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{\theta}{q}} \left( \sum_{n=1}^{k} n^{\frac{2}{p} - 1} \right)^{\frac{1}{q}}$$
\[ \geq a_{\varphi(k)} \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{q}{q+1}} \left( k^\frac{q}{q+1} - 1 \right)^{\frac{1}{q+1}} \]

\[ \geq h \left( \frac{1}{W(e^{-1})} \right)^{\frac{q}{q+1}} n^{\frac{q}{q+1}} a_{\varphi(k)}. \]

\[ \square \]

**Theorem 8.** The space \( \ell_\theta^{p,q} \) is complete for \( 1 \leq p, q \leq \infty \).

**Proof.** Let \( a^{(s)} = (a^{(s)}_n) \in \ell_\theta^{p,q} \) such that

\[ \lim_{s,t \to \infty} \| a^{(s)} - a^{(t)} \|^p \| a^{(s)} - a^{(t)} \|^q = 0. \]

For \( q = \infty \), the proof is clear. Let \( q < \infty \). Then, there exists a natural number \( s_0 \) such that

\[ \| a^{(s)} - a^{(t)} \|^p \| a^{(s)} - a^{(t)} \|^q < \eta \]

whenever \( s, t \geq s_0 \). By Lemma 3, we have

\[ a^{(s)}_k \]

\[ a^{(t)}_k \]

\[ \leq \]

\[ h \left( \frac{1}{W(e^{-1})} \right)^{\frac{q}{q+1}} \left\{ \begin{array}{ll}
q < p & \sum k^{\frac{q}{q+1} - \frac{1}{p}} \left( p R(x) \right)^{-\frac{1}{q}} k^{\frac{q}{q+1} - \frac{1}{p} \eta}, \\
p \leq q & \sum k^{\frac{q}{q+1} - \frac{1}{p} \eta} \left( p R(x) \right)^{-\frac{1}{q}} k^{\frac{q}{q+1} - \frac{1}{p} \eta}, \end{array} \right. \]

where \( h(x) = x^{-\frac{1}{q+1}} \), \( R(x) = (1 + x)^{-\frac{1}{q+1}} \). This shows that \( (a^{(s)}_k) \) is a Cauchy sequence in \( \mathbb{C} \). Thus, we have \( (a_k) \in \mathbb{C} \) such that \( \lim_{s \to \infty} |a^{(s)}_k - a_k| = 0 \). By using the equality (1) with classical method, we get \( \ell_\theta^{p,q} \) is a complete space. \[ \square \]

**Lemma 9.** Let \( 1 \leq p < \infty \), \( 1 \leq q < q_1 \leq \infty \). Then, we have the following

\[ \ell_\theta^{p,q} \subset \ell_\theta^{p,q_1}. \]

**Proof.** Let \( a = (a_n) \in \ell_\theta^{p,q} \) and \( p < q \). Then, we have by Proposition 2 in [11] that

\[ \| a \|^p_{p,q_1} = \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{q}{q+1}} \| a \|^p_{p,q_1+\varepsilon} \]

\[ \leq \sup_{0 < \varepsilon < \frac{1}{W(e^{-1})}} h(\varepsilon)^{\frac{q}{q+1}} \left( \frac{q(1 + \varepsilon)}{p} \right)^{\frac{1}{q(1 + \varepsilon)}} \| a \|^p_{p,q_1+\varepsilon} \]

\[ \leq \left( \frac{q}{p} \left( 1 + \frac{1}{W(e^{-1})} \right) \right)^{\frac{1}{q} - \frac{1}{p}} \| a \|^p_{p,q_1}, \]

\[ < \infty. \]
where \( h(x) = x^{\frac{1}{1+x}} \). The inclusion can be obtained by similar way for \( p \geq q \) with Lemma 3.

**Theorem 10.** Let either \( 1 \leq p < p_1 \leq \infty \), \( 1 \leq q < \infty \) or \( 1 \leq p < p_1 < \infty \), \( q = \infty \). Then, the inclusion

\[
\ell^\theta_{p,q} \subset \ell^\theta_{p_1,q}
\]

holds.

**Proof.** Let \( a \in \ell^\theta_{p,q} \). Then, we have

\[
\|a\|_{p_1,q},\theta = \sup_{0<\varepsilon<\frac{1}{W(\varepsilon^{-1})}} h(\varepsilon)^\frac{\theta}{p_1-q} \|a\|_{p,1+\varepsilon} \\
\leq \sup_{0<\varepsilon<\frac{1}{W(\varepsilon^{-1})}} h(\varepsilon)^\frac{\theta}{\varepsilon} \|a\|_{p,q(1+\varepsilon)} \\
= \|a\|_{p,q},\theta \\
< \infty
\]

which shows \( a \in \ell^\theta_{p_1,q} \). \( \square \)

**Corollary 11.** Let \( 1 \leq p_1 < p \leq q < q_1 \leq \infty \). Then, the inclusions

\[
\ell^{p_1},\theta \subset \ell^\theta_{p,q} \subset \ell^{q_1},\theta
\]

hold.

**Theorem 12.** The grand Lorentz sequence space \( \ell^\theta_{p,q} \) is strictly convex for \( 1 < p < \infty \) and \( 1 < q < \infty \).

**Proof.** Let \( a, b \in \ell^\theta_{p,q} \) such that \( \|a\|_{p,q},\theta = \|b\|_{p,q},\theta = 1 \) and \( \|a+b\|_{p,q},\theta = 1 \). Then, we have by using similar technique as in [1] that

\[
1 = \left\| \frac{a+b}{2} \right\|_{p,q},\theta = \sup_{0<\varepsilon<\frac{1}{W(\varepsilon^{-1})}} \varepsilon^{\frac{\theta}{p-q(1+\varepsilon)}} \left\| \frac{a+b}{2} \right\|_{p,q(1+\varepsilon)} \\
\leq \sup_{0<\varepsilon<\frac{1}{W(\varepsilon^{-1})}} \varepsilon^{\frac{\theta}{\varepsilon}} \left( \frac{\|a\|_{p,q(1+\varepsilon)} + \|b\|_{p,q(1+\varepsilon)}}{2} \right) \\
\leq \left( \frac{\|a\|_{p,q},\theta + \|b\|_{p,q},\theta}{2} \right) \\
= 1
\]

which shows \( a = b \). \( \square \)
2.2. Multiplication Operator. In this section, we characterize some properties of the multiplication operators on $\ell^\theta_{p,q}$. Let $v = (v_n)$ be a complex-valued sequence and let us define the linear transformation $M_v$ on the sequence space $X$ into the linear space of all complex-valued sequences by

$$M_v(x) = vx = (v_n x_n).$$

If the linear transformation $M_v$ is bounded with range in $X$, then it is called multiplication operator on $X$.

**Theorem 13.** Let $v = (v_n)$ be a complex-valued sequence. Then, $M_v$ is a multiplication operator on $\ell^\theta_{p,q}$ if and only if $v$ is a bounded sequence.

**Proof.** Let $M_v$ be a multiplication operator on $\ell^\theta_{p,q}$ and let $q < \infty$. Then, there exists a positive number $K > 0$ such that

$$\|M_v(a)\|_{p,q,\theta} \leq K \|a\|_{p,q,\theta}$$

for all $a \in \ell^\theta_{p,q}$. Let us define

$$e_n^{(k)} = \begin{cases} 
  s^{-\frac{\theta}{p}}, & k = n \\
  0, & k \neq n 
\end{cases}$$

where $s = \left( \frac{1}{W(e^{-1})} \right)^{\frac{1}{p+q}}$ for all $n \in \mathbb{N}$. Then, the non-increasing rearrangement of $(e_n^{(k)})$ is

$$e_{\phi(n)}^{(k)} = \begin{cases} 
  s^{-\frac{\theta}{p}}, & n = 1 \\
  0, & n \neq 1 
\end{cases}$$

Then, we have $(e_n^{(k)}) \in \ell^\theta_{p,q}$ with $\|e_n^{(k)}\|_{p,q,\theta} = 1$. By the boundedness of $M_v$, it can be written $\|M_v e_n^{(k)}\|_{p,q,\theta} \leq K \|e_n^{(k)}\|_{p,q,\theta} = K$. Thus, we get

$$\sup_{\epsilon > 0} \left( \varepsilon^\theta \sum_{n=1}^{\infty} \frac{1}{n^{1+q}} v_{\psi(n)} e_n^{(k)} \right)^{q(1+\epsilon)} n^{-1} \leq \sup_{\epsilon > 0} \left( \varepsilon^\theta \left( v_{\psi(1)} e_1^{(k)} \right)^{q(1+\epsilon)} \right)$$

which gives that $v_{\psi(1)} \leq K s^{-\frac{\theta}{p+q}}$. This shows that $v$ is bounded. If $q = \infty$, the proof is similar as was used in the classical Lorentz sequence spaces.

Conversely, let $v$ be a bounded sequence. Then, there exists $T > 0$ such that $|v_k| \leq T$ for all $k \in \mathbb{N}$. Thus, we get

$$\|M_v a\|_{p,q,\theta} = \sup_{\epsilon > 0} \left( \varepsilon^\theta \sum_{k=1}^{\infty} \left( k^{1+q} v_{\psi(k)} a_{\psi(k)} \right)^{q(1+\epsilon)} k^{-1} \right)^{\frac{1}{1+q+\epsilon}}$$
\[ \sup_{k \in \mathbb{N}} k^\frac{1}{q} \psi(k) a_{\psi(k)} \leq T \| a \|_{p,q}, \theta \]

for \( q < \infty \). If \( q = \infty \), then

\[ \sup_{k \in \mathbb{N}} k^\frac{1}{q} \psi(k) a_{\psi(k)} \leq T \| a \|_{p,q}, \theta. \]

\[ \begin{align*}
\text{Theorem 14.} & \quad \text{Let } M_v \text{ be a multiplication operator on } \ell^0_{p,q}, \ 1 \leq p, q \leq \infty. \text{ Then, } M_v \text{ is invertible if and only if there exists } \mu > 0 \text{ such that } |v_n| \geq \mu s^{-\frac{q}{p} + \frac{2q}{r}}, \text{ where } s = \left( \frac{1}{W(c^{-1})} \right)^{1/W(c-1)} \text{ for all } n \in \mathbb{N}. \\
\text{Proof.} & \quad \text{Let } M_v \text{ be invertible operator on } \ell^0_{p,q}, \ 1 \leq p, q \leq \infty. \text{ Then, there exists } \mu > 0 \text{ such that } \| M_v a \|_{p,q}, \theta \geq \mu \| a \|_{p,q}, \theta \text{ for all } a \in \ell^0_{p,q}. \text{ Thus, for } (e_n^{(k)}) \in \ell^0_{p,q}, \text{ we get}
\| M_v e_n^{(k)} \|_{p,q}, \theta = s^{-\frac{q}{p} + \frac{2q}{r}} |v_k| \geq \mu s^{-\frac{q}{p}},
\end{align*} \]

which gives \(|v_k| \geq s^{-\frac{q}{p} + \frac{2q}{r}} \mu\). Conversely, let define \( z_k = (v_k)^{-1} \). By using Theorem 5, the proof can be obtained.

\[ \begin{align*}
\text{Theorem 15.} & \quad \text{Let } M_v \text{ be a multiplication operator on } \ell^0_{p,q}, \ 1 \leq p, q \leq \infty. \text{ Then, a necessary and sufficient condition for } M_v \text{ to have closed range is that for some } \varrho > 0
\| v_n \| \geq \varrho
\text{ for each } n \in R = \{ n \in \mathbb{N} : v_n \neq 0 \}. \\
\text{Proof.} & \quad \text{Assume that } |v_n| \geq \varrho \text{ for } \varrho > 0 \text{ and for all } n \in R. \text{ Let } q < \infty \text{ and let } g^{(k)}, g \in \ell^0_{p,q} \text{ such that } M_v g^{(k)} \rightarrow g \text{ as } k \rightarrow \infty. \text{ Then, we write}
\lim_{m,n \rightarrow \infty} \| M_v g^{(m)} - M_v g^{(n)} \|_{p,q}, \theta = 0.
\end{align*} \]

Put \( x^{(mn)} = g^{(m)} - g^{(n)}. \) Thus, we have
\[ \left\{ l \in \mathbb{N} : |x_l^{(mn)}| > \frac{r}{\varrho} \right\} \subseteq \left\{ l \in \mathbb{N} : |v_l x_l^{(mn)}| > r \right\} \]

for each \( r > 0 \) and so \( g x^{(mn)} \leq v_{\psi(l)} x^{(mn)} \), where \( x^{(mn)}_{\phi(l)} \) and \( v_{\psi(l)} x^{(mn)}_{\psi(l)} \) are the non-increasing rearrangement of the sequences \( x^{(mn)}_l \) and \( v_{\psi(l)} x^{(mn)}_{\psi(l)} \), respectively.
Thus, we have
\[
\|v_x((mn))\|_{p,q,\theta} = \left\| M_v g^{(m)} - M_v g^{(n)} \right\|_{p,q,\theta}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{N}} \left( l^{1+\theta} v_\psi(l) x_\phi(l) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
\geq \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{N}} g^{q(1+\varepsilon)} \left( l^{1+\theta} x_\phi(l) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{N}} \left( l^{1+\theta} x_\phi(l) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{N}} \left( l^{1+\theta} x_\phi(l) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{N}} \left( l^{1+\theta} x_\phi(l) \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]

Since \( \|v_x((mn))\|_{p,q,\theta} \to 0 \) as \( m, n \to \infty \), we have \( x^{(mn)} \to 0 \) as \( m, n \to \infty \). This means that \( g^{(m)} \) is a Cauchy sequence in \( \ell^\theta_{p,q} \), where
\[
\ell^\theta_{p,q} = \left\{ a = (a_k) \in \ell^\theta_{p,q} : a_k = 0 \text{ if } k \in \mathbb{N} \backslash \mathbb{R} \right\}
\]
is a closed subspace of \( \ell^\theta_{p,q} \). Thus, we get \( f \in \ell^\theta_{p,q} \) such that \( g^{(m)} \to f \) as \( m \to \infty \). Since \( M_v \) is bounded linear operator, we can write \( M_v g^{(m)} \to M_v f \). This gives \( M_v f = g \). Because of \( \text{Ker} (M_v) = \ell^\theta_{p,q} \mathbb{N} \backslash \mathbb{R}, M_v \) has closed range.

Conversely, assume that \( M_v \) has closed range and there exists \( (l_n) \in \mathbb{R} \) such that \( |v_{l_n}| < \frac{1}{n} \). Let
\[
e^{(l_n)} = \begin{cases} s^{-\frac{\theta}{q}}, & m = l_n \\ 0, & m \neq l_n \end{cases}
\]
where \( s = \left( \frac{1}{1+\theta} \right) \frac{w(e-1)}{w(e-1)} \) and let \( q < \infty \). Then, \( \|e^{(l_n)}\|_{p,q,\theta} = 1 \). Thus, we get
\[
\left\| M_v e^{(l_n)} \right\|_{p,q,\theta} = \left\| e^{(l_n)} \right\|_{p,q,\theta}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{m=1}^\infty \left( m^{1+\theta} v_\psi(m) e^{(l_n)} e^{(l_m)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{m=1}^\infty \left( m^{1+\theta} v_\psi(m) e^{(l_n)} e^{(l_m)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{m=1}^\infty \left( m^{1+\theta} v_\psi(m) e^{(l_n)} e^{(l_m)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{m=1}^\infty \left( m^{1+\theta} v_\psi(m) e^{(l_n)} e^{(l_m)} \right)^{q(1+\varepsilon)} \right)^{\frac{1}{1+\theta}}
\]
\[
= \left\| e^{(l_n)} \right\|_{p,q,\theta}
\]
which means \( M_v \) is not bounded different from zero. Thus, \( |v_{l_n}| \geq \theta \) for some \( \theta > 0 \) and all \( n \in \mathbb{R} \). For the case \( q = \infty \) the proof can be obtained by similar way.

Theorem 16. Let \( M_v \) be a multiplication operator on \( \ell^\theta_{p,q} \). Then, \( M_v \) is compact if and only if \( |v_{l_n}| \to 0 \) as \( n \to \infty \).
Proof. The proof can be obtained by the similar way used in the classical Lorentz sequence space. \hfill \Box

Corollary 17. Let $M_v$ be a multiplication operator on $l_{p,q}^\theta$. Then, $M_v$ is Fredholm if and only if the set $\mathbb{N}\setminus R$ has finite elements and there exists $\rho > 0$ such that
\[ |v_n| \geq \rho \]
for all $n \in \mathbb{N}$, where $R = \{ n \in \mathbb{N} : v_n \neq 0 \}$.

References


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