





Investigating A Quadratic Bezier Curve Due to N-C-W and N-Bishop Frames

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ABSTRACT. The purpose of our paper is to investigate N-Bishop frame of the quadratic Bezier curve which is one of the effective methods for computer-aided geometric design (CAGD). Then the N-Bishop curvatures and derivative formulas for quadratics Bezier curve are calculated and give some numeric examples.

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1. INTRODUCTION

Bishop frame was developed by L. Bishop in 1975 in order to build a more practical alternative parallel frame without using the second derivative on the curve [1]. Recently, many studies have begun to be done in order to investigate the invariants of the curve via Bishop frame. The Bishop frame has been investigated by many researchers for various special curves such as involute evolute curves, Bertrand curves, helix, slant helix until now. Moreover, S. Yılmaz and his colleagues created a new Bishop alternative frame included the binomial vector instead of the tangent vector, [11]. S. Yılmaz and et. al. investigated new spherical indicators and some characterizations according to the new type Bishop frame, [11, 12]. Much work has been done on the Bishop frame as defined by L. Bishop and S. Yılmaz. Furthermore, a new alternative frame $\{N, C, W\}$ was defined by Scofield providing a different approach [9]. Uzunoglu and et. al. used a different approach to curves of constant precession [10]. Then N-Bishop frame was firstly given by Ö. Keskin. et al. by using the new Bishop frame methods in [5]. On the other hand, Pierre Bézier (1910-1999) and a French mathematician Paul de Faget de Casteljaou (1930) developed a new design curve called Bézier curves between the years 1958 and 1960. Bezier curves provide simplicity in the design processes because of their controllable usage. There are many articles about Bezier curves. The curvature and torsion of Bezier curves, Casteljaou algorithm and some other basics of Bezier curves should be found in the books [2] and [6]. Particularly, Sapidis and his colleagues studied on the curvature of a quadratic Bezier curve [8]. Moreover, Floater examined the derivatives of a Bezier curve [3]. Also, İncesu and Kusak Samancı studied on the curvatures and Bishop frame of the Bezier curves [4, 7]. The aim of this paper is to investigate a quadratic Bezier curve according to N-Bishop Frame. Firstly, we calculated Serret-Frenet frames of a quadratic Bezier curve, then firstly investigated on an alternative modified frame $\{N, C, W\}$. As a result, we computed N-Bishop frame of a quadratic Bezier curve. At the end of our study, we gave some numeric examples.

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2. PRELIMINARIES

As we know from differential geometry, the derivative formula of Serret-Frenet frame in the matrix form can be written by

$$T'(s) = \kappa N(s), \quad N'(s) = -\kappa T(s) + \tau B(s), \quad B'(s) = -\tau B(s),$$

where κ and τ are curvatures of the Serret-Frenet frame in Euclidean three space. Furthermore, the Bishop frame is relatively parallel the unit tangent field T , and the derivative matrix of the Bishop frame is given by

$$T'(s) = k_1 N_1(s) + k_2 N_2(s), \quad N_1'(s) = -k_1 T(s), \quad N_2'(s) = -k_2 T(s),$$

where k_1 and k_2 are curvatures of the Bishop frame in Euclidean 3-space, [1]. A new Bishop frame named with type-2 Bishop frame, which the frame is parallel to binomial vector field, developed in 2010 by Yılmaz et al., and its derivative matrix is described by

$$N_1'(s) = -k_1 B(s), \quad N_2'(s) = -k_2 B(s), \quad B'(s) = k_1 N_1(s) + k_2 N_2(s),$$

where k_1 and k_2 are curvatures of the type-2 Bishop frame in Euclidean 3-space, [11]. Additionally, another new alternative moving frame along a curve called $\{N, C, W\}$ frame in Euclidean 3-space was defined by Scofield in 1995, [9, 10]. The new frame is $\{N, C, N \times C = W\}$, where

$$N, C = \frac{N'}{\|N'\|}, \quad W = \frac{\tau T + \kappa B}{\|\kappa^2 + \tau^2\|}.$$

The derivative matrix of the alternative moving frame is given with

$$N'(s) = f(s)C(s), \quad C'(s) = -f(s)N(s) + g(s)W(s), \quad W'(s) = -g(s)C(s),$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and $g = \frac{\kappa^2(\tau/\kappa)'}{\kappa^2 + \tau^2} = \sigma f$ are the differentiable functions, [9, 10]. In 2017, Keskin and Yayli first defined a new kind of Bishop frame named with N-Bishop frame for a normal direction curve which is defined as an integral curve of the principal normal of a curve. The derivative matrix of the N-Bishop frame is computed by

$$N'(s) = k_1 N_1(s) + k_2 N_2(s), \quad N_1'(s) = -k_1 N(s), \quad N_2'(s) = -k_2 N(s),$$

see in [5]. On the other hand, Bezier curves were firstly developed by P. Bezier (1910-1999) and P. Casteljau (1930) between the years 1958 and 1960. A Bezier curve is defined by

$$Q(t) = \sum_{i=0}^n Q_i B_i^n(t),$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are Bernstein polynomials and Q_0, Q_1, \dots, Q_n are the control points for each $t \in [0, 1]$.

The derivatives of the Bezier curves are calculated with

$$\frac{d^r b^n}{dt^r}(t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} \Delta^r b_i \cdot B_i^{n-r}(t),$$

where $\Delta^r b_i = \sum_{j=0}^r \binom{r}{j} (-1)^{r-j} b_{i+j}$. A Bezier curve which is defined by the starting point $t = 0$ and finishing point $t = 1$ are tangent to the control polygon at these endpoints. The endpoints of the Bezier curve achieve to manipulate all of the control points. Hence the starting and finishing points are enough to control the whole curve. Many studies have been done about the Bezier curves used in the computer modeling. The basic concepts of these curves can be seen in [2, 4, 6, 7]. Moreover, the de Casteljau algorithm in Farin 1996 and Marsh 2006 provides a technique to evaluate all the points on a Bezier curve that corresponds to the parameter value $t \in [0, 1]$. In the Casteljau algorithm, the intermediate points can be found as

$$Q_{i,k}(t) = \sum_{j=0}^k B_{j,k}(t) Q_{i+j},$$

and the following properties are satisfied

$$Q_{i,0} = Q_i, \quad Q_{0,n}(t) = Q(t), \quad Q_{i,k}(t) = (1-t)Q_{i,k-1} + tQ_{i+1,k-1}.$$

The intermediate control points in Casteljau algorithm are

$$\begin{matrix} Q_{0,0} & Q_{1,0} & \cdots & Q_{n-1,0} & Q_{n,0} \\ Q_{0,1} & Q_{1,1} & \cdots & Q_{n-1,1} & \\ \vdots & & & & \\ Q_{0,n-1} & Q_{1,n-1} & & & \\ Q_{0,n} & & & & \end{matrix}$$

The first, second and third derivative of the points in Casteljau algorithm can be shown with

$$Q'(t) = n_i(Q_{1,n-1}(t) - Q_{0,n-1}(t)), \tag{2.1}$$

$$Q''(t) = n(n-1)(Q_{0,n-2}(t) - 2Q_{1,n-2}(t) + Q_{2,n-2}(t)). \tag{2.2}$$

Then the equations $Q'(0) = (Q_1 - Q_0)$, $Q'_1 = (Q_n - Q_{n-1})$ are satisfied. Additionally, the curvature and torsion of Bezier curve with n. degree can be found in [1, 3].

$$\kappa(t) = \frac{(n-1) \left\| (Q_{1,n-2}(t) - Q_{0,n-2}(t)) \times (Q_{2,n-2}(t) - Q_{1,n-2}(t)) \right\|}{n \left\| Q_{1,n-1}(t) - Q_{0,n-1}(t) \right\|^3},$$

$$\tau(t) = \frac{(n-2) \left\| (Q_{1,n-3}(t) - Q_{0,n-3}(t)) \times (Q_{2,n-3}(t) - Q_{1,n-2}(t)) \right\|}{n \left\| (Q_{1,n-2}(t) - Q_{0,n-2}(t)) \times (Q_{2,n-2}(t) - Q_{1,n-2}(t)) \right\|^2}.$$

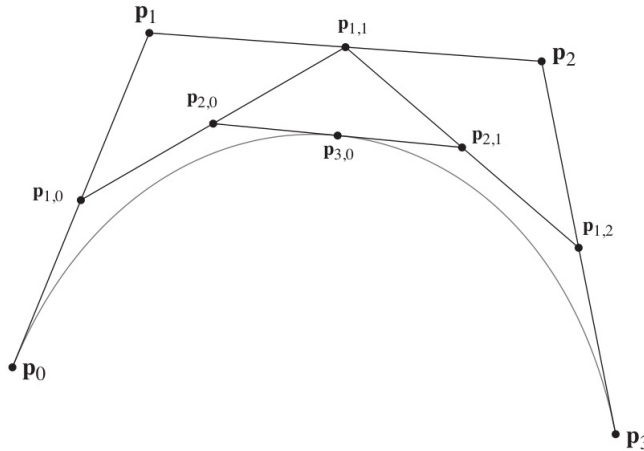


Figure 1. Casteljau Algorithm, [2].

3. MAIN RESULTS

Until now, the Serret-Frenet frame and curvatures of the Bezier curve were known. Presently, a new modified frame of a quadratic Bezier curve will be investigated in this section.

Definition 3.1. Suppose that $Q(t)$ be a quadratic Bezier curve with control points Q_0, Q_1, Q_2 . A quadratic Bezier curve is defined as

$$Q(t) = \sum_{i=0}^2 B_{i,2}(t)Q_i = (1-t)^2Q_0 + 2t(1-t)Q_1 + t^2Q_2.$$

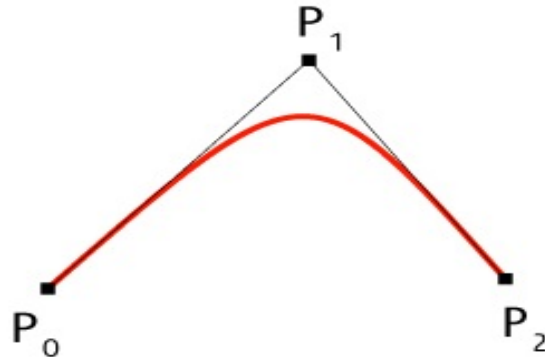


Figure 2. A quadratic Bezier curve.

Theorem 3.2. Let $Q(t)$ be a quadratic Bezier curve. The first order derivative of a quadratic Bezier curve is computed by

$$Q'(t) = 2 \left((1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2} \right).$$

Proof. Using the Eq.(2.1), we get

$$\begin{aligned} Q'(t) &= 2 \cdot (Q_{1,1}(t) - Q_{0,1}(t)) \\ &= 2 \cdot ((1-t)Q_1 + tQ_2) - [(1-t)Q_0 + tQ_1] \\ &= 2((1-t)(Q_1 - Q_0) + t(Q_2 - Q_1)) \\ &= 2 \left((1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2} \right). \end{aligned}$$

Hence the proof is completed. □

Theorem 3.3. Assume that $Q(t)$ be a quadratic Bezier curve with control points Q_0, Q_1, Q_2 . The second order derivative of a quadratic Bezier curve is obtained by $Q''(t) = 2 \left(\overrightarrow{Q_1Q_0} + \overrightarrow{Q_1Q_2} \right)$.

Proof. Using the Eq.(2.2), we obtain

$$\begin{aligned} Q''(t) &= 2 \cdot (Q_{0,0}(t) - 2Q_{1,0}(t) + Q_{2,0}(t)) = 2 \cdot (Q_0 - 2Q_1 + Q_2) \\ &= 2((Q_0 - Q_1) + (Q_2 - Q_1)) = 2 \left(\overrightarrow{Q_1Q_0} + \overrightarrow{Q_1Q_2} \right). \end{aligned}$$

Hence the proof is completed. □

Theorem 3.4. Let $Q(t)$ be a quadratic Bezier curve and Q_0, Q_1, Q_2 be control points of the quadratic Bezier curve. The Serret-Frenet frame of a quadratic Bezier curve is calculated as

$$\begin{aligned} T &= \frac{(1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2}}{\left\| (1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2} \right\|}, \\ N &= \frac{\left(\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2} \right) \times \left((1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2} \right)}{\left\| \overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2} \right\| \cdot \left\| (1-t) \overrightarrow{Q_0Q_1} + t \overrightarrow{Q_1Q_2} \right\|}, \\ B &= \frac{\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}}{\left\| \overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2} \right\|}. \end{aligned}$$

Proof. Using the Serret-Frenet frame formulas of a non-unit speed curve, the tangent vector of a quadratic Bezier curve is computed by

$$\begin{aligned} T &= \frac{Q'(t)}{\|Q'(t)\|} = \frac{(Q_{1,2-1}(t) - Q_{0,2-1}(t))}{\|Q_{1,2-1}(t) - Q_{0,2-1}(t)\|} = \frac{(Q_{1,1}(t) - Q_{0,1}(t))}{\|Q_{1,1}(t) - Q_{0,1}(t)\|} \\ &= \frac{2(((1-t)Q_1 + tQ_2) - ((1-t)Q_0 + tQ_1))}{\|2(((1-t)Q_1 + tQ_2) - ((1-t)Q_0 + tQ_1))\|} = \frac{(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}}{\|(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}\|}. \end{aligned}$$

The binormal vector of a quadratic Bezier curve is found by

$$\begin{aligned} B &= \frac{Q'(t) \times Q''(t)}{\|Q'(t) \times Q''(t)\|} \\ &= \frac{n^2(n-1)(Q_{1,n-2} - Q_{0,n-2}) \times (Q_{2,n-2} - Q_{1,n-2})}{\|n^2(n-1)(Q_{1,n-2} - Q_{0,n-2}) \times (Q_{2,n-2} - Q_{1,n-2})\|} \\ &= \frac{(Q_{1,0} - Q_{0,0}) \times (Q_{2,0} - Q_{1,0})}{\|(Q_{1,0} - Q_{0,0}) \times (Q_{2,0} - Q_{1,0})\|} = \frac{(Q_1 - Q_0) \times (Q_2 - Q_1)}{\|(Q_1 - Q_0) \times (Q_2 - Q_1)\|} \\ &= \frac{\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}}{\|\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}\|}. \end{aligned}$$

The principal normal can be calculated easily from the vector product $N = B \times T$. As a result, from above theorems, the Serret-Frenet frame is obtained. Hence the proof is completed. \square

Theorem 3.5. Let $Q(t)$ be a quadratic Bezier curve with Q_0, Q_1, Q_2 control points. The alternative $\{N, C, W\}$ frame of a quadratic Bezier curve is calculated as

$$\begin{aligned} N &= \frac{(\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}) \times ((1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2})}{\|\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}\| \cdot \|(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}\|}, \\ C &= -\frac{(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}}{\|(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}\|}, \quad W = \frac{\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}}{\|\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}\|}. \end{aligned}$$

Proof. The torsion of the quadratic Bezier curve is zero. Thus, using the Serret-Frenet frame formulas of a non-unit speed curve, the alternative frame of a quadratic Bezier curve is computed by

$$C = \frac{N'}{\|N'\|} = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}} = \frac{-\kappa T}{\sqrt{\kappa^2}} = -T.$$

Then from the tangent vector, we have

$$C = -\frac{(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}}{\|(1-t)\overrightarrow{Q_0Q_1} + t\overrightarrow{Q_1Q_2}\|},$$

and also because of

$$W = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}} = \frac{\kappa B}{\sqrt{\kappa^2}} = B,$$

then we get

$$W = \frac{\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}}{\|\overrightarrow{Q_0Q_1} \times \overrightarrow{Q_1Q_2}\|}.$$

Hence the proof is completed. \square

Corollary 3.6. Let f and g be curvatures of the alternative $\{N, C, W\}$ frame for the quadratic Bezier curve $Q(t)$. The curvatures of the modified frame can be computed as

$$f = \frac{1/4 \left\| \overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right\|}{\left\| (1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right\|^3}, \text{ and } g = 0.$$

Proof. Using the torsion of quadratic Bezier curve is zero. The curvatures will be

$$f = \sqrt{\kappa^2 + \tau^2} = |\kappa| \text{ and } g = \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{\kappa^2 + \tau^2} = 0.$$

Thus we obtain

$$f = |\kappa| = \frac{(2-1) \left\| (Q_{1,0} - Q_{0,0}) \times (Q_{2,0} - Q_{1,0}) \right\|}{2 \left\| Q_{1,1} - Q_{0,1} \right\|^3} = \frac{1}{4} \frac{\left\| \overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right\|}{\left\| (1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right\|^3}.$$

Hence the proof is completed. □

Theorem 3.7. Let Q_0, Q_1, Q_2 be the control points of a quadratic Bezier curve $Q(t)$. The alternative N-Bishop frame of a quadratic Bezier curve is calculated as

$$\begin{aligned} N_1 &= -\cos r \cdot \frac{(1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2}}{\left\| (1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right\|} - \sin r \cdot \frac{\overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2}}{\left\| \overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right\|}, \\ N_2 &= -\sin r \cdot \frac{(1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2}}{\left\| (1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right\|} + \cos r \cdot \frac{\overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2}}{\left\| \overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right\|}, \\ N &= \frac{\left(\overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right) \times \left((1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right)}{\left\| \overrightarrow{Q_0 Q_1} \times \overrightarrow{Q_1 Q_2} \right\| \cdot \left\| (1-t) \overrightarrow{Q_0 Q_1} + t \overrightarrow{Q_1 Q_2} \right\|}. \end{aligned}$$

Proof. Using the transformation matrix of N-Bishop and $\{N, C, W\}$ frame $N_1 = \cos \theta \cdot C - \sin \theta \cdot W, N_2 = \sin \theta \cdot C + \cos \theta \cdot W, N$, where $\theta = \int_0^t g(t) dt = \int_0^t 0 dt = r = \text{const.}$, the N-Bishop frame can be calculated easily. The curvatures of N-Bishop frame are $k_1 = f \cdot \cos \theta = |\kappa| \cos r, k_2 = f \cdot \sin \theta = |\kappa| \sin r.$ □

4. NUMERIC EXAMPLES

Example 4.1. Let be $Q_0 = (0, 0), Q_1 = (k, l), Q_2 = (a, 0)$ the control points of a quadratic Bezier curve which a beginning point has located at origin. Then the formula of a quadratic Bezier curve can be written as $Q(t) = \sum_{i=0}^2 B_{i,2}(t) Q_i = ((2t - 2t^2)k + t^2 a, (2t - 2t^2)l)$ from using Def. 2.6. The first, second and third derivatives are found by

$$P'(t) = ((2 - 4t)k + 2ta, (2 - 4t)l), \quad P''(t) = (-4k + 2a, -4l) \quad P'''(t) = \vec{0}.$$

Thus the torsion of the quadratic Bezier curve will be zero. The curvature is obtained by

$$\kappa = \frac{-al/4}{[(-k - tk + ta - tk)^2 + (l - 2tl)^2]^{3/2}}.$$

The Serret-Frenet frame is

$$\begin{aligned} T &= \frac{1}{\mu} (2k - 4tk + 2ta, 2l - 4tl), \\ N &= \frac{1}{\mu} (4tl - 2l, 2k - 4tk + 2ta), \\ B &= (0, 0, 1), \end{aligned}$$

where $\mu = \sqrt{(2k - 4tk + 2ta)^2 + (2l - 4tl)^2}$. The $\{N, C, W\}$ frame is found by

$$\begin{aligned} N &= \frac{1}{\mu} (4tl - 2l, 2k - 4tk + 2ta), \\ C &= -\frac{1}{\mu} (2k - 4tk + 2ta, 2l - 4tl), \\ W &= (0, 0, 1). \end{aligned}$$

As a result, the N-Bishop frame is obtain by

$$\begin{aligned} N_1 &= \left(-\frac{\cos r}{\mu} (2k - 4tk + 2ta), -\frac{\cos r}{\mu} (2l - 4tl), -\sin r \right) \\ N_2 &= \left(-\frac{\sin r}{\mu} (2k - 4tk + 2ta), \frac{\sin r}{\mu} (2l - 4tl), \cos r \right) \\ N &= \frac{1}{\mu} (4tl - 2l, 2k - 4tk + 2ta). \end{aligned}$$

Example 4.2. Let be $Q_0 = (0, 0)$, $Q_1 = (0, 1)$, $Q_2 = (1, 0)$ the the control points of a quadratic Bezier curve, see in [2]. Then $Q(t) = \sum_{i=0}^2 B_{i,2}(t)Q_i = (t^2, 2t - 2t^2)$. Thus the torsion of the quadratic Bezier curve will be zero. The curvature is obtained by $\kappa = \frac{-1/4}{[t^2 + (1-2t)^2]^{3/2}}$. The Serret-Frenet frame is $T = \frac{1}{\mu} (2t, 2 - 4t)$, $N = \frac{1}{\mu} (4t - 2, 2t)$, $B = (0, 0, 1)$ where $\mu = \sqrt{2t^2 + (2 - 4t)^2}$. The modified frame is calculated by $C = -\frac{1}{\mu} (2t, 2 - 4t)$, $W = (0, 0, 1)$, $N = \frac{1}{\mu} (4t - 2, 2t)$. The N-Bishop frame is obtained as

$$N_1 = \left(-\frac{2 \cos r}{\mu} t, -\frac{\cos r}{\mu} (2 - 4t), -\sin r \right), \text{ and } N_2 = \left(-\frac{2 \sin r}{\mu} t, \frac{\sin r}{\mu} (2 - 4t), \cos r \right), N = \frac{1}{\mu} (4t - 2, 2t).$$

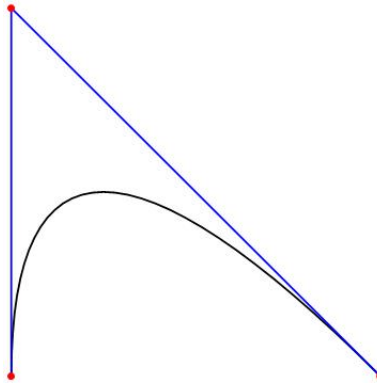


Figure 3. A Mathematica output of the Example 4.2.

5. CONCLUSION

The concept of a quadratic Bezier curve method is one of the important methods for CAGD. In our paper, some properties of a quadratic Bezier curve have been calculated for according to a new alternative frame called N-Bishop frame. Some results of the N-Bishop frame have the abilities to be computed an alternative curvature and torsion for all points of the quadratic Bezier curves. Thus we think that our study will guide other researchers.

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CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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