GENERALIZED FUZZY SUBHYPERSPACES BASED ON FUZZY POINTS

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Abstract. We define \((\in, \in \vee q)\)-fuzzy subhyperspaces and \((\in, \in \vee q_k)\)-fuzzy subhyperspaces, as a generalization of fuzzy subhyperspaces, \((\in, \in \vee q)\)-fuzzy subhyperspaces and \((\in, \in \vee q_k)\)-fuzzy subhyperspaces. In this way, we show that \((\in, \in \vee q_k)\)-fuzzy subhyperspaces are the largest family of generalized fuzzy subhyperspaces based on concepts of belongingness and quasi-coincidence. Moreover, we study some properties and investigate the difference of generalized fuzzy subhyperspaces, supported by examples.

1. Introduction

The theory of fuzzy set was initiated by Zadeh [23] in 1965. It was extended to algebra by Rosenfeld [17] with defining fuzzy subgroups. Then other fuzzy algebraic structures have been investigated, such as fuzzy semigroups, fuzzy ideals, fuzzy vector spaces and so on. For more information about fuzzy algebraic structures refer to [15] and [16].

Algebraic hyperstructures was introduced by Marty [14] in 1934, when he defined hypergroups. Similarly fuzzy algebraic hyperstructures were investigated in many branches [8]. Ameri [1] introduced fuzzy subhyperspaces of hypervector spaces in the sense of Scafatti-Tallini [18]. Fuzzy subhyperspaces were studied more in [2], [3] and [13].

After defining the concept of \((\in, \in \vee q)\)-fuzzy subgroups by Bhakat and Das [4] as an important generalization of Rosenfeld’s fuzzy subgroups, this notion and another type, \((\in, \in \vee q_k)\)-fuzzy subgroups, were studied on many algebraic structures (see [9]). In context of hyperstructures theory, as an extension of fuzzy subhyperstructures, Davvaz and Corsini defined \((\in, \in \vee q)\)-fuzzy subhyperquasigroups in [5]. Furthermore, semihypergroups were characterized by \((\in, \in \vee q_k)\)-fuzzy hyperideals and \((\in, \in \vee q_k)\)-fuzzy hyperideals in [19] and [20], respectively. Moreover,
(ε, ∈ ∨q)-fuzzy n-ary subhypergroups in [12] and [22], (ε, ∈ ∨q)-fuzzy and (ε, ∨q, ∨q)-fuzzy n-ary subpolygroups in [11] and (ε, ∈, ∈ ∨q)-fuzzy n-ary subhypergroups in [21], had been studied. Also, this concept and related topics were investigated on Hv-rings in [6], hypermodules in [24] and (m, n)-ary hypermodules in [7].

A new generalization of (ε, ∈ ∨q)-fuzzy subgroups was defined by Jun et al. ([10]) which called (ε, ∈ ∨q)-fuzzy subgroups. Now, in this paper, we introduce new generalizations of a fuzzy subhyperspace. In this regards, (ε, ∈ ∨q)-fuzzy subhyperspaces and (ε, ∈ ∨qk)-fuzzy subhyperspaces as generalizations of fuzzy subhyperspaces, (ε, ∈ ∨q)-fuzzy subhyperspaces and also (ε, ∈ ∨qk)-fuzzy subhyperspaces are defined. It is shown that these notions construct a bigger family for generalized fuzzy subhyperspaces and also indicated that subhyperspaces are characterized by them. Moreover, connections and differences of them are studied, supported by illustrative examples.

2. Preliminaries

In this section we present some definitions and properties of hypervector spaces and fuzzy subhyperspaces that we shall use in later.

Definition 1. [18] Let K be a field, (V, +) be an Abelian group and Ps(V) be the set of all non-empty subsets of V. We define a hypervector space over K to be the quadruplet (V, +, ○, K), where “○” is an external hyperoperation

\[ \circ : K \times V \rightarrow Ps(V), \]

such that for all \( a, b \in K \) and \( x, y \in V \) the following conditions hold:

\[ (H_1) \quad a \circ (x + y) \subseteq a \circ x + a \circ y, \text{ right distributive law}, \]
\[ (H_2) \quad (a + b) \circ x \subseteq a \circ x + b \circ x, \text{ left distributive law}, \]
\[ (H_3) \quad a \circ (b \circ x) = (ab) \circ x, \]
\[ (H_4) \quad a \circ (-x) = (-a) \circ x = -(a \circ x), \]
\[ (H_5) \quad x \in 1 \circ x, \]

where in (H1), \( a \circ x + a \circ y = \{ p + q : p \in a \circ x, q \in a \circ y \} \). Similarly it is in (H2). Also in (H3), \( a \circ (b \circ x) = \bigcup_{t \in bx} a \circ t \).

V is called strongly right distributive, if we have equality in (H1). In a similar way we define the strongly left distributive hypervector spaces. V is called strongly distributive, if it is strongly right and left distributive.

A non-empty subset W of V is called a subspace of V if W is itself a hypervector space with the external hyperoperation on V, i.e. for all \( a \in K \) and \( x, y \in W, x - y \in W \) and \( a \circ x \subseteq W \).

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified.
Example 2. [2] In classical vector space \((\mathbb{R}^3, +, 0, \mathbb{R})\) we define:
\[
\begin{align*}
o & : \mathbb{R} \times \mathbb{R}^3 \rightarrow P_+(\mathbb{R}^3) \\
o(a, (x_0, y_0, z_0)) &= L,
\end{align*}
\]
where \(L\) is a line with the parametric equations:
\[
L : \begin{cases}
x = ax_0, \\
y = ay_0, \\
z = t.
\end{cases}
\]
Then \(V = (\mathbb{R}^3, +, 0, \mathbb{R})\) is a strongly left distributive hypervector space.

Definition 3. [1] A fuzzy subset \(\mu\) of \(V\) is called a fuzzy subhyperspace of \(V\), if for all \(a \in K\) and \(x, y \in V\), the following conditions are satisfied:
1) \(\mu(x - y) \geq \mu(x) \land \mu(y)\),
2) \(\bigwedge_{t \in \mathbb{R}} \mu(t) \geq \mu(x)\).

Example 4. (modified example 2.16, of [3]) Consider the hypervector space \(V = (\mathbb{R}^3, +, 0, \mathbb{R})\) in Example 2. Define a fuzzy subset \(\mu\) of \(V\) by the following:
\[
\mu(x, y, z) = \begin{cases}
t_3 & (x, y, z) \in \{0\} \times \{0\} \times \mathbb{R}, \\
t_2 & (x, y, z) \in \mathbb{R} \times \{0\} \times \mathbb{R} \setminus \{0\} \times \{0\} \times \mathbb{R}, \\
t_1 & \text{otherwise},
\end{cases}
\]
where \(0 \leq t_1 < t_2 < t_3 \leq 1\). Then \(\mu\) is a fuzzy subhyperspace of \(V\).

3. \((\alpha, \beta)\)-Fuzzy Subhyperspaces

A fuzzy subset \(\mu\) of a hypervector space \(V\) defined by
\[
\mu(y) = \begin{cases}
t \neq 0, & \text{if } y = x \\
0, & \text{if } y \neq x
\end{cases}
\]
is said to be a fuzzy point with the support \(x\) and the value \(t\) and is denoted by \(x_t\). For a fuzzy point \(x_t\) and the fuzzy subset \(\mu\) we write
1) \(x_t \in \mu \Leftrightarrow \mu(x) \geq t\),
2) \(x_tq\mu \Leftrightarrow \mu(x) + t > 1\),
3) \(x_tq_k\mu \Leftrightarrow \mu(x) + t + k > 1\), for \(k \in [0, 1]\),
4) \(x_tq^\delta\mu \Leftrightarrow \mu(x) + t + \delta > 1\), for \(\delta \in (0, 1]\),
5) \(x_tq^\beta_k\mu \Leftrightarrow \mu(x) + t + k > \delta\), for \((k, \delta) \in [0, 1) \times (0, 1]\).

In case (1) we say that \(x_t\) is belong to \(\mu\) and in (2) \(x_t\) is quasi-coincident with the fuzzy subset \(\mu\). For a fuzzy point \(x_t\), we write \(x_t \in \lor\mu\) \((x_t \in \lor\mu\) if \(x_t \in \mu\) or \(x_tq\mu\) \((x_t \in \mu\) and \(x_tq\mu\)). Similarly, we have \(x_t \in \lor\mu\) and \(x_t \in \lor\mu\). Also, for \(\alpha \in \{\lor, \lor, \lor, \lor, ..., \}\), the notation \(x_t\alpha\mu\) means that \(x_t\alpha\mu\) does not hold.

Definition 5. A fuzzy subset \(\mu\) of \(V\) is called an \((\alpha, \beta)\)-fuzzy subhyperspace of \(V\), if for all \(t, r \in [0, 1]\), \(x, y \in V\) and \(\alpha \in K\):
\[\kappa \beta 1) x_t \in \mu \text{ and } y_r \in \mu \text{ imply that } (x - y)_{t \lor r} \in \lor^\beta\mu;\]
Similarly, Example 8.

Thus

Then

Proof. Let \( \mu \) be an \((\varepsilon, \in \ast V_k)\)-fuzzy subhyperspace of \( V \). Assume that \((k\delta 1)\) is not valid, i.e., there exist \( x, y \in V \) such that \( \mu(x - y) < \mu(x) \land \mu(y) \land \delta - k \). Then \( \mu(x - y) < t \leq \mu(x) \land \mu(y) \land \delta - k \), for some \( t \in (0, 1] \). Thus \( t \leq \mu(x) \) and \( t \leq \mu(y) \), and so \( x_t, y_t \in \mu \). Hence \( (x - y)_t \in V_k^\mu \). But \( \mu(x - y) < t \) and also \( \mu(x - y) + t < t + t \leq \delta - k + \delta - k = \delta - k \). It follows that \( \mu(x - y) + t + \delta \leq \delta \). Therefore, \( (x - y)_t \in V_k^\mu \), which is a contradiction. Consequently, \((k\delta 1)\) is valid.

Now if there exist some \( x, z \in V \) and \( a \in K \), such that \( z \in a \ast x \) and \( \mu(z) < \mu(x) \land \delta - k \), then \( \mu(z) < t \leq \mu(x) \land \delta - k \), for some \( t \in (0, 1] \). Thus \( t \leq \mu(z) \) and \( x_t \in \mu \). Hence \( z_t \in V_k^\mu \). But \( \mu(z) < t \) and also \( \mu(z) + t < t + t \leq \delta - k + \delta - k = \delta - k \). Thus \( \mu(z) + t + k \leq \delta \). Therefore \( z_t \in V_k^\mu \), which is a contradiction. Consequently, \((k\delta 2)\) is valid.

Conversely, let \( x_t \in \mu \) and \( y_t \in \mu \). Then \( \mu(x) \geq t \) and \( \mu(y) \geq r \). Thus by \((k\delta 1)\), \( \mu(x - y) \geq t \land r \land \delta - k \). If \( t \land r \leq \delta - k \), then \( \mu(x - y) \geq t \land r \) and so \((x - y)_{t \land r} \in \mu \).

Corollary 7. Let \( \mu \in FS(V) \), i.e. \( \mu \) is a fuzzy subset of \( V \). Then \( \mu \) is

1) an \((\varepsilon, \in \ast V_k)\)-fuzzy subhyperspace of \( V \) if and only if for all \( x, y \in V \) and \( a \in K \), \( \mu(x - y) \geq \mu(x) \land \mu(y) \land 0.5 \) and \( \bigwedge_{z \in a \ast x} \mu(z) \geq \mu(x) \land 0.5 \);

2) an \((\varepsilon, \in \ast V_k)\)-fuzzy subhyperspace of \( V \) if and only if for all \( x, y \in V \) and \( a \in K \), \( \mu(x - y) \geq \mu(x) \land \mu(y) \land \delta - k \) and \( \bigwedge_{z \in a \ast x} \mu(z) \geq \mu(x) \land \delta - k \);

3) an \((\varepsilon, \in \ast V_k)\)-fuzzy subhyperspace of \( V \) if and only if for all \( x, y \in V \) and \( a \in K \), \( \mu(x - y) \geq \mu(x) \land \mu(y) \land \delta - k \) and \( \bigwedge_{z \in a \ast x} \mu(z) \geq \mu(x) \land \delta - k \).

Example 8. Consider the hypervector space \( V = (\mathbb{R}^3, +, o, \mathbb{R}) \) in Example 2. Define a fuzzy subset \( \mu \) of \( V \) by the following:

\[
\mu(x, y, z) = \begin{cases} 
\frac{1}{2} \frac{x}{1-x}, & (x, y, z) = (0, 0, 0), \\
0, & (x, y, z) \in \mathbb{R} \times \{0\} \times \{0\} \setminus \{0, 0, 0\}, \\
\mu, & \text{o.w.}
\end{cases}
\]
Then $\mu$ is an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$, but it is not an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{5}{2}})$-fuzzy subhyperspace of $V$, since the condition $(k\delta 2)$ is not valid (if $x = (0, 0, 0)$ and $z = (0, 0, 2)$, then for all $a \in \mathbb{R}$, $z \in a \circ x$, so $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq (\mu(x) \lor \frac{\delta - k}{2})$).

In the next example one can see that an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace is not an $(\varepsilon, \in \mathcal{V}_n^{d+\frac{5}{2}})$-fuzzy subhyperspace or $(\varepsilon, \in \mathcal{V}_q^{d+\frac{5}{2}})$-fuzzy subhyperspace or $(\varepsilon, \in \mathcal{V}_n^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$, in general.

**Example 9.** Consider the $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace $\mu$ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:

1. $\mu$ is not an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$, because if $x = (0, 0, 0)$ and $z = (0, 0, 3)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq (\mu(x) \lor \frac{\delta - k}{2})$;
2. $\mu$ is not an $(\varepsilon, \in \mathcal{V}_n^{d+\frac{3}{2}})$-fuzzy subhyperspace of $V$, because if $x = (0, 0, 0)$ and $z = (0, 0, 2)$, then for all $a \in \mathbb{R}$ and $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq (\mu(x) \lor \frac{1 - k}{2})$;
3. $\mu$ is not an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{5}{2}})$-fuzzy subhyperspace of $V$, because if $x = (1, 0, 0)$, $a = 2$ and $z = (2, 0, 5)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{2}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq (\mu(x) \lor \frac{\delta - k}{2})$.

**Theorem 10.** Let $\mu \in \mathcal{F}_S(V)$, $\delta \in (0, 1]$ and $k \in [0, 1)$. Then $\mu$ is an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$ if and only if $\mu_t(\neq \emptyset)$ is a subhyperspace of $V$, for all $t \in (0, \frac{\delta - k}{2})$.

**Proof.** Suppose $\mu$ is an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$, $t \in (0, \frac{\delta - k}{2})$ and $x, y \in \mu_t$. Then by Theorem $[6]$

$$\mu(x - y) \geq \mu(x) \lor \mu(y) \lor \frac{\delta - k}{2} \geq t \lor \frac{\delta - k}{2} = t.$$  

Thus $x - y \in \mu_t$. Moreover, for all $a \in K$, $z \in a \circ x$ and $x \in \mu_t$, we have $\mu(z) \geq \mu(x) \lor \frac{\delta - k}{2} \geq t$, which means that $a \circ x \subseteq \mu_t$. Hence $\mu_t$ is a subhyperspace of $V$, for all $t \in (0, \frac{\delta - k}{2})$.

Conversely, let $\mu_t$ be a subhyperspace of $V$, for all $t \in (0, \frac{\delta - k}{2})$ and let $(k\delta 1)$ is not valid. Then there exist $x, y \in V$ and $t \in (0, 1)$ such that

$$\mu(x - y) < t < \mu(x) \lor \mu(y) \lor \frac{\delta - k}{2}.$$  

Thus $x, y \in \mu_t$ for some $0 < t \leq \frac{\delta - k}{2}$, but $x - y \not\in \mu_t$, which is a contradiction. Hence $(k\delta 1)$ is valid. Similarly, we can show $(k\delta 2)$ is valid. Therefore, by Theorem $[6]$, $\mu$ is an $(\varepsilon, \in \mathcal{V}_q^{d+\frac{7}{2}})$-fuzzy subhyperspace of $V$. 

**Corollary 11.** Let $\mu \in \mathcal{F}_S(V)$. Then $\mu$ is

1. an $(\varepsilon, \in \mathcal{V}_q)$-fuzzy subhyperspace of $V$ if and only if $\mu_t(\neq \emptyset)$ is a subhyperspace of $V$, for all $t \in (0, 0.5)$;
2) an \((\varepsilon, \in \cap q_k)\)-fuzzy subhyperspace of \(V\) if and only if \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\), for all \(t \in (0, \frac{1-k}{2}]\).

3) an \((\varepsilon, \in \cap q^\delta)\)-fuzzy subhyperspace of \(V\) if and only if \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\), for all \(t \in (0, \frac{\delta}{2}]\).

**Theorem 12.** Let \(\mu \in FS(V)\). Then \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\), for all \(t \in (\frac{\delta-k}{2}, 1]\), if and only if

(i) \(\mu(x - y) \lor \frac{\delta-k}{2} \geq \mu(x) \land \mu(y)\), for all \(x, y \in V\);

(ii) \(\bigwedge_{z \in \cap q_{afx}} \mu(z) \lor \frac{\delta-k}{2} \geq \mu(x)\), for all \(x \in V\) and \(a \in K\).

**Proof.** Let \(\mu_t(\neq \emptyset)\) be a subhyperspace of \(V\), for all \(t \in (\frac{\delta-k}{2}, 1]\). If there exist \(x, y \in V\) such that

\[\mu(x - y) \lor \frac{\delta-k}{2} < \mu(x) \land \mu(y),\]

then \(t_0 = \mu(x) \land \mu(y) \in (\frac{\delta-k}{2}, 1]\) and \(x, y \in \mu_{t_0}\). Thus \(x - y \in \mu_{t_0}\) and so \(\mu(x - y) \geq t_0\), which is a contradiction with \(\mu(x - y) \lor \frac{\delta-k}{2} < t_0\). Hence (i) holds. Similarly, condition (ii) will be obtained.

Conversely, assume that \(t \in (\frac{\delta-k}{2}, 1]\) and \(x, y \in \mu_t\). Then

\[\mu(x - y) \lor \frac{\delta-k}{2} < \mu(x) \land \mu(y) \geq t > \frac{\delta-k}{2}.\]

Thus \(\mu(x - y) \geq t\) and so \(x - y \in \mu_t\). Now let \(a \in K\), \(x \in \mu_t\) and \(z \in a \circ x\). Then

\[\mu(z) \lor \frac{\delta-k}{2} \geq \bigwedge_{z \in \cap q_{afx}} \mu(z) \lor \frac{\delta-k}{2} \geq \mu(x) \geq t,\]

which implies that \(\mu(z) \geq t\), for all \(z \in a \circ x\). Hence \(a \circ x \subseteq \mu_t\), for all \(t \in (\frac{\delta-k}{2}, 1]\). Therefore, \(\mu_t\) is a subhyperspace of \(V\).

**Corollary 13.** Let \(\mu \in FS(V)\). Then

1) \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\) for all \(t \in (0.5, 1]\) if and only if for all \(x, y \in V\) and \(a \in K\), \(\mu(x - y) \lor 0.5 \geq \mu(x) \land \mu(y)\) and \(\bigwedge_{z \in \cap q_{afx}} \mu(z) \lor 0.5 \geq \mu(x)\);

2) \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\) for all \(t \in (\frac{1-k}{2}, 1]\) if and only if for all \(x, y \in V\) and \(a \in K\), \(\mu(x - y) \lor \frac{1-k}{2} \geq \mu(x) \land \mu(y)\) and \(\bigwedge_{z \in \cap q_{afx}} \mu(z) \lor \frac{1-k}{2} \geq \mu(x)\);

3) \(\mu_t(\neq \emptyset)\) is a subhyperspace of \(V\) for all \(t \in (\frac{\delta}{2}, 1]\) if and only if for all \(x, y \in V\) and \(a \in K\), \(\mu(x - y) \lor \frac{\delta}{2} \geq \mu(x) \land \mu(y)\) and \(\bigwedge_{z \in \cap q_{afx}} \mu(z) \lor \frac{\delta}{2} \geq \mu(x)\).

**Theorem 14.** A non-empty subset \(S\) of \(V\) is a subhyperspace of \(V\) if and only if \(\chi_S\) is an \((\varepsilon, \in \cap q_k)\)-fuzzy subhyperspace of \(V\).
Proof. Let $S$ be a subhyperspace of $V$ and $t \in (0, \frac{\delta-k}{2}]$. If $x, y \in \chi_{S_1}$, then $\chi_S(x), \chi_S(y) \geq t$. Thus $\chi_S(x) = \chi_S(y) = 1$ and so $x - y \in S$. Hence $\chi_S(x - y) = 1 \geq t$, i.e. $x - y \in \chi_S$. Similarly, $a \circ x \subseteq \chi_{S_1}$, for all $a \in K$ and $x \in \chi_{S_1}$. Consequently, $\chi_{S_1}$ is a subhyperspace of $V$, for all $t \in (0, \frac{\delta-k}{2}]$. Therefore, by Theorem 10 $\chi_S$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$.

Conversely, let $\chi_S$ be an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$, $a \in K$ and $x \in S$. Then by Theorem 6, for all $z \in a \circ x$, \[ \bigwedge_{z \in a \circ x} \chi_S(z) \geq \chi_S(x) \wedge \frac{\delta-k}{2} = 1 \wedge \frac{\delta-k}{2} = \frac{\delta-k}{2}. \]

Since $\delta \in (0,1]$ and $k \in [0,1)$, so $\chi_S(z) = 1$, for all $z \in a \circ x$. Thus $a \circ x \subseteq S$. Similarly, $x + y \in S$, for all $x, y \in S$. Therefore, $S$ is a subhyperspace of $V$. □

It is well-known that the characterization function of any subhyperspace is a fuzzy subhyperspace. Hence the following corollary is obtained from Theorem 14.

Corollary 15. A non-empty subset $S$ of $V$ is a subhyperspace of $V$ if and only if $\chi_S$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ if and only if $\chi_S$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ if and only if $\chi_S$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$.

Proposition 16. Let $\delta \in (0,1]$, $k \in [0,1)$. Then

1) Every $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$, if $\delta + k < 1$;
2) Every $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$, if $\delta + k > 1$;
3) For $\delta + k = 1$, $\mu$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ if and only if it is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$;
4) Every $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$, if $\delta = k + 1$;
5) Every $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$ is a fuzzy subhyperspace of $V$, if $\delta = k$.

Proof. 1) Let $\mu$ be an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$. Then by Corollary 7(2), for all $x, y \in V$ and $a \in K$, it follows that:

$$\mu(x - y) \geq \mu(x) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \frac{\delta}{2},$$

and

$$\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \wedge \frac{1-k}{2} \geq \mu(x) \wedge \frac{\delta}{2}.$$ 

Thus by Corollary 7(3), $\mu$ is an $(\varepsilon, \in \cap q^k)$-fuzzy subhyperspace of $V$.

2) The proof is completed by Corollary 7 similarly.
3) One can conclude by Corollary 7(2) and Corollary 7(3).
4) It is straightforward by Theorem 6 and Corollary 7(1).
5) The proof is obtained by Theorem 6 and Definition 3. □

The following example shows that the converse of assertions (1) and (2) of Proposition 16 are not generally true.
Example 17. Consider the fuzzy subset $\mu$ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8. Then by Corollary 7, it follows that:

1) $\mu$ is an $(\varepsilon, \in \vee 0.2)$-fuzzy subhyperspace of $V$, but it is not an $(\varepsilon, \in \vee 0.4)$-fuzzy subhyperspace of $V$, because if $x = (2, 0, 0), a = 4$ and $z = (8, 0, 3)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}$, $\mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq \mu(x)^{\frac{1.8}{2}} (\delta = 0.2, k = 0.4)$.

2) $\mu$ is an $(\varepsilon, \in \vee 0.7)$-fuzzy subhyperspace of $V$, but it is not an $(\varepsilon, \in \vee 0.5)$-fuzzy subhyperspace of $V$, because if $x = (3, 0, 0), a = 1$ and $z = (3, 0, 2)$, then $z \in a \circ x$, $\mu(x) = \frac{1}{3}, \mu(z) = \frac{1}{5}$ and $\mu(z) \not\geq \mu(x)^{\frac{3.5}{2}} (\delta = 0.5, k = 0.7)$.

Theorem 18. If $\mu$ is an $(\varepsilon, \in \vee 0.7)$-fuzzy subhyperspace of $V$, such that $\mu(x) \leq \frac{\delta - k}{2}$, for all $x \in V$, then $\mu$ is a fuzzy subhyperspace of $V$.

Proof. By Theorem 6, for all $x, y \in V$ and $a \in K$, $\mu(x - y) \geq \mu(x) \land \mu(y) \land \frac{\delta - k}{2} = \mu(x) \land \mu(y)$ and $\bigwedge_{z \in a \circ x} \mu(z) \geq \mu(x) \land \frac{\delta - k}{2} = \mu(x)$. So $\mu$ is a fuzzy subhyperspace of $V$.

Note that in the $(\varepsilon, \in \vee 0.7)$-fuzzy sub hyperspace $\mu$ of $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 8, $\mu(x) \not\leq \frac{\delta - k}{2}$, for some $x \in V$ and $\mu$ is not a fuzzy subhyperspace of $V$.

Corollary 19. Let $\mu \in FS(V)$. Then

1) If $\mu$ is an $(\varepsilon, \in \vee 0.4)$-fuzzy subhyperspace of $V$, such that $\mu(x) < 0.5$, for all $x \in V$, then $\mu$ is a fuzzy sub hyperspace of $V$;

2) If $\mu$ is an $(\varepsilon, \in \vee 0.2)$-fuzzy subhyperspace of $V$, such that $\mu(x) < \frac{1.8}{2}$, for all $x \in V$, then $\mu$ is a fuzzy subhyperspace of $V$;

3) If $\mu$ is an $(\varepsilon, \in \vee 0.5)$-fuzzy subhyperspace of $V$, such that $\mu(x) < \frac{3.5}{2}$, for all $x \in V$, then $\mu$ is a fuzzy subhyperspace of $V$.

Proposition 20. Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If $\mu$ is an $(\varepsilon, \in \vee 0.7)$-fuzzy subhyperspace of $V$, then it is an $(\varepsilon, \in \vee 0.5)$-fuzzy subhyperspace of $V$.

Proof. By Theorem 6, $\mu(x - y) \geq \mu(x) \land \mu(y) \land \frac{\delta_1 - k}{2}$ and $\mu(z) \geq \mu(x) \land \frac{\delta_1 - k}{2}$, for all $x, y \in V$, $a \in K$ and $z \in a \circ x$. Since $\delta_1 \geq \delta_2$, thus $\mu(x - y) \geq \mu(x) \land \mu(y) \land \frac{\delta_2 - k}{2}$ and $\mu(z) \geq \mu(x) \land \frac{\delta_2 - k}{2}$. Hence the proof is completed by Theorem 6.

In the following example it can be seen that the converse of Proposition 20 is not generally valid.

Example 21. Consider the fuzzy subset $\mu$ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then $\mu$ is an $(\varepsilon, \in \vee 8.0.8)$-fuzzy subhyperspace of $V$ and $\mu$ is not an $(\varepsilon, \in \vee 0.5)$-fuzzy subhyperspace of $V$, while $\delta_2 = 0.8 \leq \delta_1 = 0.95$.

Corollary 22. Let $0 < \delta_2 \leq \delta_1 \leq 1$ and $\mu \in FS(V)$. If $\mu$ is an $(\varepsilon, \in \vee 0.7)$-fuzzy subhyperspace of $V$, then it is an $(\varepsilon, \in \vee 0.2)$-fuzzy subhyperspace of $V$.

Proof. It is straightforward by Corollary 7 and Proposition 20.

Next example shows that the converse of Corollary 22 is not valid, in general.
**Example 23.** Consider the fuzzy subset $\mu$ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then $\mu$ is an $(\varepsilon, \in \mathcal{V}q^{0.3})$-fuzzy subspace of $V$ and $\mu$ is not an $(\varepsilon, \in \mathcal{V}q^{0.5})$-fuzzy subspace of $V$, while $\delta_2 = 0.3 \leq \delta_1 = 0.5$.

**Proposition 24.** Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If $\mu$ is an $(\varepsilon, \in \mathcal{V}q^{\delta_1})$-fuzzy subspace of $V$, then it is an $(\varepsilon, \in \mathcal{V}q^{\delta_2})$-fuzzy subspace of $V$.

**Proof.** It is completed by a similar manner of the proof of Proposition 20. The converse of Proposition 24 is not valid in general. See the following example:

**Example 25.** Consider the fuzzy subset $\mu$ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then $\mu$ is an $(\varepsilon, \in \mathcal{V}q^{0.3})$-fuzzy subspace of $V$ and $\mu$ is not an $(\varepsilon, \in \mathcal{V}q^{0.5})$-fuzzy subspace of $V$, while $k_1 = 0.2 \leq k_2 = 0.4$.

The following corollary is immediately concluded by Corollary 7 and Proposition 21.

**Corollary 26.** Let $0 \leq k_1 \leq k_2 < 1$ and $\mu \in FS(V)$. If $\mu$ is an $(\varepsilon, \in \mathcal{V}q_{k_1})$-fuzzy subspace of $V$, then it is an $(\varepsilon, \in \mathcal{V}q_{k_2})$-fuzzy subspace of $V$.

The converse of Corollary 26 is not valid in general. See the next example:

**Example 27.** Consider the fuzzy subset $\mu$ of the hypervector space $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, in Example 8. Then $\mu$ is an $(\varepsilon, \in \mathcal{V}q_{0.5})$-fuzzy subspace of $V$ and $\mu$ is not an $(\varepsilon, \in \mathcal{V}q_{0.7})$-fuzzy subspace of $V$, while $k_1 = 0.5 \leq k_2 = 0.7$.

**Theorem 28.** Let $S$ be a subspace of $V$. Then for every $t \in (0, \frac{\delta - k}{2}]$, there exists an $(\varepsilon, \in \mathcal{V}q^k)$-fuzzy subspace $\mu$ of $V$ such that $\mu_t = S$.

**Proof.** Let $t \in (0, \frac{\delta - k}{2}]$ and define a fuzzy subset $\mu$ of $V$ as

$$
\mu(x) = \begin{cases} 
0 & \text{if } x \in S \\
t & \text{otherwise}
\end{cases}
$$

Clearly, $\mu_t = S$. Now if there exist $x, y \in V$ such that $\mu(x - y) < \mu(x) \cap \mu(y) \cap \frac{\delta - k}{2}$, then $\mu(x - y) = 0$ and $\mu(x) = \mu(y) = t$, which is a contradiction. Thus $\mu(x - y) \geq \mu(x) \cap \mu(y) \cap \frac{\delta - k}{2}$, for all $x, y \in V$. Similarly, $\bigwedge_{z \in \mathcal{V}q^k} \mu(z) \geq \mu(x) \cap \frac{\delta - k}{2}$, for all $a \in K$.

Hence $\mu$ is an $(\varepsilon, \in \mathcal{V}q^k)$-fuzzy subspace of $V$, by Theorem 6.

**Corollary 29.** Let $S$ be a subspace of $V$. Then

1. For every $t \in (0, 0.5]$, there exists an $(\varepsilon, \in \mathcal{V}q)$-fuzzy subspace $\mu$ of $V$, such that $\mu_t = S$;
2. For every $t \in (0, \frac{\delta - k}{2}]$, there exists an $(\varepsilon, \in \mathcal{V}q_k)$-fuzzy subspace $\mu$ of $V$, such that $\mu_t = S$;
3. For every $t \in \left(0, \frac{\delta}{2}\right]$, there exists an $(\varepsilon, \in \mathcal{V}q^\delta)$-fuzzy subspace $\mu$ of $V$, such that $\mu_t = S$. 


Theorem 30. If $\mu_i$ is an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$, for all $i \in I$, then $\mu = \cap_{i \in I} \mu_i$ is an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$.

Proof. Let $x_t, y_t \in \mu$, for $x, y \in V$ and $t, r \in (0, 1)$ and $(x-y)_{t\wedge r} \in \vee q_{\delta}^k \mu$. Then $\mu(x-y) < t \wedge r$ and $\mu(x-y) + t \wedge r \leq \delta - k$, which imply that $\mu(x-y) < \frac{\delta - k}{2}$. Now, put $I_1 = \{i \in I \mid (x-y)_{t\wedge r} \in \mu_i\}$ and $I_2 = \{i \in I \mid (x-y)_{t\wedge r} \in \mu_i\} \cap \{j \in I \mid (x-y)_{t\wedge r} \in \mu_j\}$. Then $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. If $I_2 = \emptyset$, then $(x-y)_{t\wedge r} \in \mu_i$, for all $i \in I$, which implies that $\mu(x-y) \geq t \wedge r$, that is a contradiction. Hence $I_2 \neq \emptyset$, and so for every $i \in I_2$, $\mu_i(x-y) < t \wedge r$ and $\mu_i(x-y) + t \wedge r > \delta - k$, that is $t \wedge r > \frac{\delta - k}{2}$. Thus from $x_t, y_t \in \mu$, we can obtain $\mu_i(x) \wedge \mu_i(y) > \mu(x) \wedge \mu(y) > t \wedge r > \frac{\delta - k}{2}$. Now, set $\alpha = \mu_i(x-y) < \frac{\delta - k}{2}$ and take $\alpha < \beta < \frac{\delta - k}{2}$. Then $x_\alpha, y_\beta \in \mu_i$ but $\mu_i(x-y) = \alpha < \beta$ and $\mu_i(x-y) + \beta < \delta - k$. This contradicts $\mu_i$ is an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$. Thus $\mu_i(x-y) > \frac{\delta - k}{2}$, which is a contradiction. Hence $(x-y)_{t\wedge r} \in \vee q_{\delta}^k \mu$. Similarly, the condition $(k\delta \over 2)$ will be proven. Therefore, $\mu = \cap_{i \in I} \mu_i$ is an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$. 

For any fuzzy set $\mu$ of $V$ and $t \in (0, 1)$, we denote

$$(\mu)_t = \{x \in V \mid x_t \in q_{\delta}^k \mu\} \quad \text{and} \quad [\mu]_t = \{x \in V \mid x_t \in \vee q_{\delta}^k \mu\}.$$ 

Obviously, $[\mu]_t = \mu_t \cup (\mu)_t$.

Theorem 31. Let $\mu \in FS(V)$. Then $\mu$ is an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$ if and only if $[\mu]_t$ is a subspace of $V$, for all $t \in (0, 1)$.

Proof. Let $\mu$ be an $(\varepsilon, \varepsilon \vee q_{\delta}^k)$-fuzzy subspace of $V$ and $x, y \in [\mu]_t$, for $t \in (0, 1)$. Then $\mu(x) \geq t$ or $\mu(x) + t > \delta - k$, and $\mu(y) \geq t$ or $\mu(y) + t > \delta - k$. Using Theorem 30,

1) for $\mu(x), \mu(y) \geq t$, $\mu(x-y) \geq t \wedge \frac{\delta - k}{2}$. If $t > \frac{\delta - k}{2}$, then $\mu(x-y) + t > \frac{\delta - k}{2} + \frac{\delta - k}{2} = \delta - k$, and so $(x-y)_t \in \mu$. If $t \leq \frac{\delta - k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x-y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

2) for $\mu(x) \geq t$ and $\mu(y) + t > \delta - k$, $\mu(x-y) \geq t \wedge \delta - k - t \wedge \frac{\delta - k}{2}$. If $t > \frac{\delta - k}{2}$, then $\mu(x-y) > \delta - k - t$, and so $(x-y)_t \in \mu$. If $t \leq \frac{\delta - k}{2}$, then $\mu(x-y) \geq t$ and so $(x-y)_t \in \mu$. Thus $x-y \in [\mu]_t$. Similarly, we can show $a \circ x \subseteq [\mu]_t$, for all $a \in K$, in this case.

3) for $\mu(x) + t > \delta - k$ and $\mu(y) \geq t$, we can prove similar to the case (2).

4) for $\mu(x) + t > \delta - k$ and $\mu(y) + t > \delta - k$, if $t > \frac{\delta - k}{2}$, then $\mu(x-y) > \delta - k - t$, and if $t \leq \frac{\delta - k}{2}$, then $\mu(x-y) \geq t$. Thus $(x-y)_t \in \vee q_{\delta}^k \mu$ and so $x-y \in [\mu]_t$. Similarly, $a \circ x \subseteq [\mu]_t$, for all $a \in K$.

Therefore, $[\mu]_t$ is a subspace of $V$.

Conversely, let $\mu$ be a fuzzy subset of $V$ and there exist $x, y \in V$ such that $\mu(x - y) < \mu(x) \wedge \mu(y) + \frac{\delta - k}{2}$, for $\delta \in (0, 1)$ and $k \in (0, 1)$. Then $\mu(x - y) < t \geq \mu(x) \wedge \mu(y) + \frac{\delta - k}{2}$, for some $t \in (0, 1)$. Thus $x, y \in [\mu]_t$ and so $x-y \in [\mu]_t$. But
\[ \mu(x - y) < t \quad \text{and} \quad \mu(x - y) + t \geq \delta - k, \]
which is a contradiction. Hence (1) of Theorem [6] and similarly the assertion (2) of Theorem [6] are valid. Therefore, \( \mu \) is an \((\varepsilon, \nu V_{q_k})\)-fuzzy subhyperspace of \( V \).

4. Conclusion

We define \( \lambda(x) + t > \delta \) and \( \lambda(x) + t + k > \delta \) as new connections between a fuzzy point and a fuzzy subset on a hypervector space to generalize the concept of fuzzy subhyperspaces. These new connections help us to find new generalizations for fuzzy subhyperspaces and specially the largest family of them based on the concepts of belongingness and quasi-coincidence. This study can be extended to other algebraic structures and hyperstructures, in future. The following figure shows how we extend the family of generalized fuzzy subhyperspaces:

\[ \text{Figure 1. Generalizations of fuzzy subhyperspaces} \]

References


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