Results in Nonlinear Analysis 3 (2020) No. 1, 12–17
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

KKM implies the Brouwer fixed point theorem: Another proof

Sehie Park[a],

[a] The National Academy of Sciences, Republic of Korea; Seoul 06579, and
Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea.

Abstract

It is well-known that the Brouwer fixed point theorem (BFPT), the weak Sperner combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem are mutually equivalent and have scores of equivalent formulations and several thousand applications. It is well-known that KKM deduced the BFPT from Sperner Lemma. In this article, we recall some KKM theoretic results implying the BFPT.

Keywords: Abstract convex space; KKM space; Brouwer fixed point theorem; Sperner lemma, metric type space.
2010 MSC: 47H04, 47H10, 49J27, 49J35, 54C60, 54H25, 91B50.

1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained the so-called KKM theorem from the weak Sperner lemma and applied it to a new proof of the Brouwer fixed point theorem. Later in 1961, Fan extended the KKM theorem to any topological vector spaces and applied it to various results including the Schauder fixed point theorem.

Since then there have appeared a large number of works devoting applications of the KKM theory to Analytical Fixed Point Theory, that is, the theory mainly concerns to topological vector spaces.

In the present article, we recall some KKM theoretic results applicable to a proof of the Brouwer fixed point theorem. Actually we give some modern versions of the proof of the Brouwer theorem by applying the KKM theorem and its generalized form.

This paper is an abridged version of our forthcoming work [18].

Email address: park35@snu.ac.kr; sehiepark@gmail.com; http://parksehie.com (Sehie Park)

Received February 28, Accepted: March 15, 2020, Online: March 18, 2020.

2. Old mathematical trinity

In 1912, the following Brouwer fixed point theorem appeared:

Theorem. (Brouwer [1]) *A continuous map from an n-simplex to itself has a fixed point.*

In this theorem, an n-simplex can be replaced the unit ball $\mathbb{B}^n$ or any compact convex subset in $\mathbb{R}^n$ without affecting its conclusion.

In 1928, Sperner obtained the following combinatorial lemma and its applications:

Lemma. (Sperner [20]) *Let K be a simplicial subdivision of an n-simplex $v_0v_1\cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n)$, the number assigned to u is one of the integers $i_0, i_1, \cdots, i_k$. Then the total number of those n-simplices of K, whose vertices receive all $n+1$ integers $0, 1, \cdots, n$, is odd. In particular, there is at least one such n-simplex.*

The particular case of the above is usually called the *weak Sperner lemma.*

Indeed, using the weak Sperner lemma as a starting point, three of the greatest topologists of all times, Polish academician S. Mazurkiewicz and two of his former doctoral students, B. Knaster and K. Kuratowski published in 1929 the following so-called the KKM theorem, which is the origin of the KKM theory:

Theorem. (KKM [7]) *Let $A_i\,(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0p_1\cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1}\cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1}\cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

This is first applied to a direct proof of the Brouwer fixed point theorem by KKM in 1929, and then to a von Neumann type minimax theorem for arbitrary topological vector spaces by Sion in 1958. Later it was known that the KKM theorem also holds for open-valued KKM map; see [15].

In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications. See [15].

$$\begin{array}{ccc} & \text{Brouwer} & \\ 1974 \swarrow & & \nwarrow 1929 \\ \text{weak Sperner} & \underset{1929}{\longrightarrow} & \text{KKM} \end{array}$$

Recall that the KKM theorem follows from the Sperner lemma and is used to obtain one of the most direct proof of the Brouwer theorem. Therefore, it was conjectured that those three theorems are mutually equivalent. This was clarified by Yoseloff in 1974. In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications. Moreover, Park and Jeong [19] also gave a proof of the weak Sperner lemma from the Brouwer fixed point theorem.

Halpern [4] first introduced the outward and, later, inward sets:

Let E be a t.v.s. and $X \subset E$. The *inward and outward sets* of X at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined by as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X-x), \quad O_X(x) = x + \bigcup_{r<0} r(X-x).$$

Recall that a map $f : X \to E$ is said to be

weakly inward if $f(x) \in \overline{I_X(x)}$ for each $x \in \mathrm{Bd}\ X$.
weakly outward if $f(x) \in \overline{O_X(x)}$ for each $x \in \mathrm{Bd}\ X$.

In [9], we deduced a generalization or an equivalent form of the Brouwer fixed point theorem by applying the Fan-Browder fixed point theorem, which is equivalent to the KKM theorem.

Theorem (Park [9]) *Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a weakly inward [resp. outward] functions such that*

$$\{x \in X : \operatorname{Re} p(x) < \operatorname{Re} p(f(x))\}$$

is open for all $p \in E^*$. Then f has a fixed point.

Our aim in this paper is to give a variant of the proof of the fact that KKM implies the Brouwer fixed point theorem.

3. Abstract convex spaces

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition. [10]-[12] Let E be a topological space, D a nonempty set, and $\Gamma : \langle D \rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples of abstract convex spaces are given, for example, in [10]-[12], [16], [17].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{or} \quad \Gamma_A \subset F^+ G(A) \text{ for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ to a set Z is called *a $\mathfrak{K}$-map* and we say that F belongs to *the KKM family* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our previous works, we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the (partial) KKM principle. See [10]-[12], [16] and the references therein.

We already obtained the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}$$
$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longleftrightarrow \phi_A\text{-space}$$
$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

Recall that any simplex is a KKM space by the KKM theorem and its open-valued version, and that any convex subset of a t.v.s. is a KKM space by the proof of the 1961 KKM Lemma of Ky Fan; see [3]. For other subclasses of (partial) KKM spaces in the diagram, all proofs were well-established in the literature; see [12]-[15].

4. KKM maps in metric type spaces

We introduce a more general concept than metric spaces introduced by Khamsi and Hussain [6] as follows:

Definition. Let M be a set. Let $\delta : M \times M \to [0, \infty)$ be a function which satisfies

(1) $\delta(x, y) = 0$ if and only if $x = y$;

(2) $\delta(x, y) = \delta(y, x)$ for any $x, y \in M$;

(3) $\delta(x, y) \leq k(\delta(x, z) + \delta(z, y))$ for any points $x, y, z \in M$, for some constant $k > 0$.

Then the pair (M, δ) is called a *metric type space*.

In [6], examples of metric type spaces are given and, for metric type spaces, the concepts of convergence, completeness, openness, closedness, closure, topology, compactness, totally boundedness, and others are defined as usual. Moreover, the following are defined in [6]:

Let A be a nonempty bounded subset of a metric type space (M, δ). Then we define as follows:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap\{B \subset M \mid B \text{ is a closed ball in } M \text{ such that } A \subset B\}$.

(ii) $\mathcal{A}(M) := \{A \subset M \mid A = \mathrm{ad}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

(iii) A is called *subadmissible*, if for each $N \in \langle A\rangle$, $\mathrm{ad}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

For an $x \in M$ and $\varepsilon > 0$, let

$$B(x, \varepsilon) := \{y \in M \mid \delta(x, y) \leq \varepsilon\} \quad \text{and} \quad N(x, \varepsilon) := \{y \in M \mid \delta(x, y) < \varepsilon\}.$$

It is amazing that, in metric type spaces, when we do not know whether open balls are open and closed balls are closed; see [6].

We introduce new definitions:

Definition. An abstract convex space $(M, D; \Gamma)$ is called simply a *metric type space* if (M, δ) is a metric type space, $D \subset M$ is a nonempty subset, and $\Gamma : \langle D\rangle \to \mathcal{A}(M)$ is a map such that $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle D\rangle$. A map $G : D \multimap M$ is a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D\rangle$.

A Γ-convex subset of $(M \supset D; \Gamma)$ is said to be *subadmissible*.

Remark. 1. For a metric space M, $(M \supset D; \Gamma)$ is given in [4], where $\Gamma_A := \mathrm{ad}(A)$. This is a metric type space.

2. Let M be a metric space and D a nonempty set. For each $A := \{a_0, a_1, \ldots, a_n\} \in \langle D\rangle$, choose a subset $B := \{x_0, x_1, \ldots, x_n\} \in \langle M\rangle$ and define $\Gamma_A := \mathrm{ad}(B)$. Then $(M, D; \Gamma)$ is not a metric type space. For this space, the so-called generalized $gKKM$ mapping in [2] is not a KKM map.

In [14], we obtained a Schauder type fixed point theorem for metric type spaces:

Theorem 3.1. *Let $(M \supset D; \Gamma)$ be a metric type space and X a Γ-convex subset of M such that $X \cap D$ is dense in X. If the identity map $1_X \in \mathfrak{KC}(X, X)$ [resp. $1_X \in \mathfrak{KO}(X, X)$], then any compact continuous function $f : X \to X$ has a fixed point.*

5. KKM maps in normed vector spaces

From Theorem 3.1, we obtain new proofs of the Schauder and Brouwer fixed point theorems.

Recall the following well-known Schauder conjecture in 1935; see *The Scottish Book* [7], Problem 54.

Conjecture (Schauder) *Every nonempty compact convex subset X of a (metrizable) t.v.s. E has the fixed point property; that is, every continuous map $f : X \to X$ has a point $x_0 \in X$ such that $x_0 = f(x_0)$.*

This is not resolved yet. Here we give a partial solution as follows:

Theorem 4.1. *Let E be a metrizable t.v.s. and X a Γ-convex subset of M. If the identity map $1_X \in \mathfrak{KC}(X, X)$ [resp. $1_X \in \mathfrak{KO}(X, X)$], then any compact continuous function $f : X \to X$ has a fixed point.*

Here X is a Γ-convex whenever, for any $A \in \langle X \rangle$, we have $\Gamma_A := \mathrm{BI}(A) \subset X$.

Proof. Put $M = D = E$ in Theorem 3.1. □

From Theorem 3.1 or Theorem 4.1, we have the following form of the Schauder fixed point theorem:

Theorem 4.2. *Let M be a normed vector space, $X := B(O, r)$ be a closed ball of M with center the origin O. Then any compact continuous function $f : X \to X$ has a fixed point.*

Proof. For any $A \in \langle X \rangle$, let $\Gamma_A := \mathrm{BI}(A)$. Since there is a closed ball in X containing A, we have $\Gamma_A \subset X$. For any two points $x, y \in X$, the line segment $\overline{xy} \subset \mathrm{BI}(\{x, y\}) \subset X$. Hence X is a convex subset of a t.v.s. and a KKM space. Therefore, by Theorem 3.1, any compact continuous function $f : X \to X$ has a fixed point. □

From Theorem 4.2, we immediately have the following:

Theorem 4.3. *Let $\mathbb{R}^n$ be a Euclidean space, $\mathbb{B}^n := B(O, 1)$ be a closed ball with center the origin O. Then any continuous function $f : \mathbb{B}^n \to \mathbb{B}^n$ has a fixed point.*

Since any homeomorphic image of a space having the fixed point property has the same property, we have the following Brouwer fixed point theorem:

Theorem 4.4. (Brouwer [1]) *A continuous map from an n-simplex to itself has a fixed point.*

References

[1] L.E.J. Brouwer, *Über Abbildung von Mannigfaltigkeiten*, Math. Ann. **71** (1912) 97–115.
[2] C.-M. Chen and T.-H. Chang, *Some results for the family $2\text{-}_gKKM(X, Y)$ and the Φ-mapping in hyperconvex metric spaces*, Nonlinear Anal. **69** (2008) 2533–2540.
[3] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Annalen **142** (1961) 305–310.
[4] B. Halpern, *Fixed-point theorems for outward maps*, Doctoral Thesis, U.C.L.A. 1965.
[5] M.A. Khamsi, *KKM and Ky Fan theorems in hyperconvex metric spaces*, J. Math. Anal. Appl. **204** (1996) 298–306.
[6] M.A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **73** (2010) 3123–3129. doi:10.1016/j.na.2010.06.084.
[7] B. Knaster, K. Kuratowski und S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe*, Fund. Math. **14** (1929) 132–137.
[8] R.D. Mauldin, The Scottish Book: Mathematics from the Scottish Café, Birkhäuser, Boston-Basel-Stuttgart, 1981.
[9] S. Park, *A generalization of the Brouwer fixed point theorem*, Bull. Korean Math. Soc. **28** (1991) 33–37.
[10] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008) 1–27.
[11] S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex. Anal. **9**(3) (2008) 331–350.
[12] S. Park, *The KKM principle in abstract convex spaces — Equivalent formulations and applications*, Nonlinear Analysis **73** (2010) 1028–1042.
[13] S. Park, *Fixed point theorems in the new era of the KKM theory*, Fixed Point Theory and Its Applications (Proc. ICFPTA-2009), 145–159, Yokohama Publ., 2010.
[14] S. Park, *Comments on the KKM theory of metric type spaces*, Linear and Nonlinear Anal. **2**(1) (2016) 39–45.
[15] S. Park, *A history of the KKM Theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **56**(2) (2017) 1–51.
[16] S. Park, *On further examples of partial KKM spaces*, J. Fixed Point Theory 2019:10 (18 June, 2019), 1–18.
[17] S. Park, *Extending the realm of Horvath spaces*, J. Nonlinear Convex Anal. **20**(8) (2019) 1609–1621.
[18] S. Park, *KKM implies the Brouwer fixed point theorem: Revisited*, to appear.

[19] S. Park and K.S. Jeong, *A proof of the Sperner lemma from the Brouwer fixed point theorem*, Nonlinear Anal. Forum **8** (2003), 65–67.

[20] E. Sperner, *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abh. Math. Seminar Univ. Hamburg **6** (1928) 265–272.