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BERTRAND PARTNER CURVES ACCORDING TO DARBOUX FRAME IN THE EUCLIDEAN 3-SPACE E^3

F. KAYA AND A. YILDIRIM

Abstract. In this study, Bertrand partner curves are researched. Characterization of Bertrand partner curves lying on different or the same two surfaces are examined according to Darboux frame and some special curves that we obtain with them .

1. Introduction

The interest of special curves has increased recently. One of the best example for these curves is Bertrand partner curves. J. Bertrand introduced this curves in 1850[4]. If principal normal vector field of two curves collides with each other, these curves are called as Bertrand partner curves. Any Bertrand curve

$$\begin{array}{lllll} \alpha: & I\subset R & \longrightarrow & E^3 & \\ & s & \longrightarrow & \alpha(s)= & (\alpha_1(s),\alpha_2(s),\alpha_3(s)) \end{array}$$

is defined by the equation $\lambda\kappa(s)+\mu\tau(s)=1$. While λ, μ are constant, κ and τ are curvature and torsion of the curve α, respectively[2].

Many researches have been conducted about Bertrand partner curves. Some of them are Kazaz and his friendsfriends[6], İlarslan - Aslan[5] and Masal-Azak[7].

In this study, Bertrand partner curves are analyzed by using Darboux frame. We will research the states of geodesic curve, asymptotic curve and principle line of Bertrand partner curves.

2. Preliminaries

Let the curve $\alpha: \ I\subset R \longrightarrow E^3$ be a regular curve specified by arc parameter s. The triadic $\{T,N,B\}$ obtained as indicated below by means of this curve is called as Frenet vector field at the $\alpha(s)$ point of the curve α.

$$T(s)= \alpha'(s),$$

$$N(s)= \frac{\alpha''(s)}{\|\alpha''(s)\|},$$

$$B(s)= T(s)\times N(s).$$

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In this point, $\times$ indicates vector product on E^3 and $\alpha'(s) = \dfrac{d\alpha}{ds}$.

These vectors form an orthogonal vector system. In that point, whereas T is called as tangent vector field, N and B are called as principles normal vector field and binormal vector field, respectively. The Frenet formulas could be given in the figure below:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Here κ and τ are called as curvature and torsion of the α curve, respectively.

If the curve α is over a surface S, another frame which is called as Darboux frame and showed with $\{T, g, n\}$, could be found. In this frame, T is tangent vector of the curve α, n is unit normal vector field of the surface S and finally g is another vector field written with $g = n \times T$. Relation between the Frenet frame and the Darboux frame could be given in the figure below:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}. \tag{2.1}$$

Here θ is the angle between g and N. Derivative change of the Darboux frames are as shown below:

$$\begin{bmatrix} T' \\ g' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}.$$

Here κ_g , κ_n and τ_g in the equation above are called as geodesic curve, normal curve and geodesic torsion of the curve α, respectively. The relationship between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given in the figure below.

$$\kappa_g = \kappa\cos\theta, \quad \kappa_n = \kappa\sin\theta, \quad \tau_g = \tau + \theta'.$$

For the curve α in surface S, we can say the followings[6].

- **i.** α is a geodesic curve if and only if $\kappa_g = 0$,
- **ii.** α is an asymptotic line if and only if $\kappa_n = 0$,
- **iii.** α is a principle curvature line if and only if $\tau_g = 0$.

Definition 2.1. Let E^3 be the 3-dimensional Euclidean space with the standard inner product. Let two regular curves be (α, α^*) in E^3 and also the Frenet vectors of the curves α and α^* be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. If $\{N, N^*\}$ is linearly dependent, in that case the partner curves (α, α^*) are called as Bertrand partner curves.

In this study, we are going to use $\{T, N, B, \kappa, \tau, s\}$ as Frenet elements of the curve α and $\{T^*, N^*, B^*, \kappa^*, \tau^*, s^*\}$ as Frenet elements of the curve α^*. In this point, s and s^* are arc parameter of the curve α and arc parameter of the curve α^*, respectively.

Theorem 2.2. [1] *Let (α, α^*) be Bertrand partner curves. There is equilibrium indicated below between those partner curves.*

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)N(s)$$

$\lambda(s)$ in the equation is a constant.

Theorem 2.3. [3] *Let (α, α^*) be Bertrand partner curves. We could write the followings:*

- *The distance between their opposite points is fixed.*
- *The angle between tangents at opposite points is fixed.*
- *$\tau\tau^*$ is a constant.*
- *If the equation $\tau = 0$ is valid, every curve has infinitely Bertrand partner curves.*
- *If the equations of $\tau \neq 0$ and $\psi = \dfrac{\pi}{2}$ are correct for the Bertrand partner curves (α, α^*) specifying tangents at opposite points as the angle ψ, the curvatures of Bertrand partner curves is constant, and one is geometric location of other's curvature centers.*
- *If both κ and τ are constant, there are infinitely Bertrand partner curves which are all cyclical helix.*

Theorem 2.4. [1] *Let (α, α^*) be Bertrand partner curves. The following equations exist:*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

In this equation, ψ is the angle between T and T^*.

Theorem 2.5. [2] *Let (α, α^*) be Bertrand partner curves. We could get the equation below.*

$$\begin{aligned} \cos\psi \frac{ds^*}{ds} &= 1 - \lambda\kappa, \\ \sin\psi \frac{ds^*}{ds} &= \lambda\tau. \end{aligned} \tag{2.2}$$

Here λ is a constant.

3. Bertrand Partner Curves According to Darboux Frame in E^3

In this part, we are going to present the characterization of Bertrand partner curves by paying into consideration of Darboux frame.

Definition 3.1. Let S and S^* be directed two surface in E^3 and also the curves of α and α^* be given in the surfaces S and S^*, respectively. If the curves α and $\alpha^*$are Bertrand partner curves, these two curves are called as Bertrand partner curves in accordance with the surfaces S and S^*. They are indicated as $(S_\alpha, S^*_{\alpha^*})$.

We are going to show the Darboux elements of the curve α as $\{T, g, n, \kappa_g, \kappa_n, \tau_g, s\}$ and the Darboux elements of the curve α^* as $\{T^*, g^*, n^*, \kappa^*_g, \kappa^*_n, \tau^*_g, s^*\}$ for Bertrand pair $(S_\alpha, S^*_{\alpha^*})$. Here κ_g, κ_n, τ_g and $\kappa^*_g, \kappa^*_n, \tau^*_g$ are geodesic curvature, normal curvature, torsion curvature of the curves α and α^*, respectively; at the same time s and s^* are arc elements of the curves α and α^*, respectively.

Theorem 3.2. *Let Bertrand partner curves $(S_\alpha, S^*_{\alpha^*})$ be given. Relation between the Darboux frames are as like stated below:*

$$T^* = \cos\psi T - \sin\psi\sin\theta g - \sin\psi\cos\theta n, \tag{3.1}$$

$$\begin{aligned} g^* &= \sin\theta^*\sin\psi T + (\cos\theta^*\cos\theta + \sin\theta\sin\theta^*\cos\psi)g \\ &\quad +(-\sin\theta\cos\theta^* + \sin\theta^*\cos\theta\cos\psi)n, \end{aligned} \tag{3.2}$$

$$\begin{aligned} n^* &= \cos\theta^*\sin\psi T + (-\sin\theta^*\cos\theta + \cos\theta^*\sin\theta\cos\psi)g \\ &\quad +(\sin\theta\sin\theta^* + \cos\theta^*\cos\theta\cos\psi)n. \end{aligned} \tag{3.3}$$

In those equations, ψ, θ, θ^* are the angles between T and T^*, g and N, g^* and N^*, respectively.

Proof. We have given in the equation (2.1) that the relation of the curve α between the Frenet frame and the Darboux frame is

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}.$$

The relation of the curve α^* between the Frenet frame and the Darboux frame is

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^* & -\sin\theta^* \\ 0 & \sin\theta^* & \cos\theta^* \end{bmatrix} \begin{bmatrix} T^* \\ g^* \\ n^* \end{bmatrix}. \tag{3.4}$$

On the other side, the relation of the curves α and α^* between their Frenet frames is

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix}. \tag{3.5}$$

If we use the equation (3.5) in the equation (2.1), we obtain the equation below.

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}. \tag{3.6}$$

Finally, if we use the equation (3.4) in the equation (3.6), we could finish the proof of the theorem. □

Theorem 3.3. *The correlation between Bertrand partner curves $(S_\alpha, S^*_{\alpha^*})$ and their curvatures is:*

$$\kappa^*_g\sin\theta^* + \kappa^*_n\cos\theta^* = \frac{\cos\psi\sin\theta g + \cos\psi\cos\theta n}{1 - \lambda\kappa_g\cos\theta + \lambda\kappa_n\sin\theta}.$$

Proof. We know that the equation for the curve α is

$$N = \cos\theta g - \sin\theta n. \tag{3.7}$$

On the other side, for Bertrand partner curves $(S_\alpha, S^*_{\alpha^*})$ is

$$\alpha^* = \alpha + \lambda N \tag{3.8}$$

If we use the equation (3.7) in the equation (3.8), the equation (3.8) becomes as shown below:

$$\alpha^{*}=\alpha+\lambda\cos\theta g-\lambda\sin\theta n \tag{3.9}$$

and if we take the derivative of the equation (3.9) with respect to s and use Darboux frame, we find the equation below:

$$\begin{aligned} T^{*}\frac{ds^{*}}{ds} &= \left(1-\lambda\kappa_g\cos\theta+\lambda\kappa_n\sin\theta\right)T \\ &\quad +\lambda\sin\theta\left(\tau_g-\theta'\right)g+\lambda\cos\theta\left(\tau_g-\theta'\right)n. \end{aligned} \tag{3.10}$$

If we multiply both sides of the equation (3.10) by T and use in the equation (3.5), we obtain the equation given below:

$$\frac{ds^{*}}{ds}=\frac{1-\lambda\kappa_g\cos\theta+\lambda\kappa_n\sin\theta}{\cos\psi} \tag{3.11}$$

If we take the derivative of the equation (3.1) with respect to s, and use the Darboux frames, we get the equation below:

$$\begin{aligned} \kappa_g^{*}g^{*}+\kappa_n^{*}n^{*} &= \left(\kappa_g\sin\psi\sin\theta+\kappa_n\sin\psi\cos\theta\right)T \\ &\quad +\left(\kappa_g\cos\psi-\theta'\sin\psi\cos\theta+\tau_g\sin\psi\cos\theta\right)g \\ &\quad +\left(\kappa_n\cos\psi+\theta'\sin\psi\cos\theta-\tau_g\sin\psi\cos\theta\right)n. \end{aligned} \tag{3.12}$$

In this equation (3.12) , ψ is constant because we know from the Theorem 2.3. After multiplying both sides of the equation (3.12) by T, and then if we use in the equation (3.2) and the equation (3.3), we attain the equation below:

$$\left(\kappa_g^{*}\sin\theta^{*}\sin\psi+\kappa_n^{*}\cos\theta^{*}\sin\psi\right)\frac{ds^{*}}{ds}=\kappa_g\sin\theta\sin\psi+\kappa_n\cos\theta\sin\psi. \tag{3.13}$$

If we use the equation (3.11) in the equation (3.13), we finish the proof. □

When we consider the Theorem 3.3, we could find the corollaries below:

Corollary 3.4. Let Bertrand partner curves $(S_\alpha, S^{*}_{\alpha^{*}})$ be specified.

(1) If both of the curves α and $\alpha^{*}$are geodesic

$$\kappa_n^{*}=\frac{\cos\psi\cos\theta\kappa_n}{\cos\theta^{*}+\lambda\kappa_n\sin\theta\cos\theta^{*}}.$$

(2) If both of the curves α and α^{*} are asymptotic curves

$$\kappa_g^{*}=\frac{\cos\psi\sin\theta\kappa_g}{\cos\theta^{*}+\lambda\kappa_n\sin\theta\cos\theta^{*}}.$$

(3) If α is an asymptotic curve and α^{*} is a geodesic curve

$$\kappa_n^{*}=\frac{\cos\psi\cos\theta\kappa_g}{\cos\theta^{*}-\lambda\kappa_g\cos\theta\cos\theta^{*}}.$$

(4) If α is a geodesic curve and α^{*} is an asymptotic curve

$$\kappa_g^{*}=\frac{\cos\psi\cos\theta\kappa_g}{\sin\theta^{*}+\lambda\kappa_n\sin\theta\sin\theta^{*}}.$$

Theorem 3.5. *Let Bertrand partner curves $(S_\alpha, S^{*}_{\alpha^{*}})$ be specified. There is equation below:*

$$\tau_g=\frac{-\sin\psi+\lambda\kappa_g\cos\theta\sin\psi-\lambda\kappa_n\sin\theta\sin\psi+\lambda\theta'\cos\psi}{\lambda\cos\psi}.$$

Proof. If we use the equation (3.11) in the equation (2.2), we obtain the equation shown below.

$$\lambda\tau = \sin\psi \frac{1 - \lambda\kappa_g \cos\theta + \lambda\kappa_n \sin\theta}{\cos\psi}. \tag{3.14}$$

And if we use the equation $\tau_g = \tau + \theta'$ in the equation (3.14), we prove the theorem. □

If the Theorem 3.5 is paid attention, we could find the corollaries below:

Corollary 3.6. Let Bertrand partner curves $(S_\alpha, S^*_{\alpha^*})$ be specified.

(1) If T and T^* are linear dependent or in other worlds, if the equation is $\psi = 0$;

$$\tau_g = \theta'.$$

(2) If the angle θ between g and N is constant, and α is an asymptotic curve;

$$\tau_g = \frac{-\sin\psi + \lambda\kappa_g \cos\theta \sin\psi}{\lambda\cos\psi}.$$

(3) If α is a geodesic curve and α^* is an asymptotic curve, then the torsion curvature of α;

$$\tau_g = \frac{-\sin\psi + \lambda\theta' \cos\psi}{\lambda\cos\psi}.$$

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ehit Abdlkadir Ouz Anadolu mam Hatip Lisesi, 63200, Şanliurfa-Turkey.
E-mail address: f.kaya73@outlook.com

Harran University, Science and Arts Faculty, Mathematics Department, 63300, Şanliurfa-Turkey.
E-mail address: abdullahyildirim@harran.edu.tr