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Cauchy problem with ψ -Caputo fractional derivative in Banach spaces

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Abstract

This paper is devoted to the existence of solutions for certain classes of nonlinear differential equations involving the ψ -Caputo fractional derivative in Banach spaces. Our approach is based on a new fixed point theorem with respect to convex-power condensing operator combined with the technique of measures of noncompactness. Finally, two examples are given to illustrate the obtained results.

Keywords: ψ -Caputo fractional derivative, Cauchy problem, convex-power condensing operator, fixed point theorem, Banach spaces, measures of noncompactness.

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1. Introduction

Fractional differential equations gained considerable importance due to their various applications in different fields of science such as physics, chemistry, economics, polymer rheology, aerodynamics, electrodynamics of complicated medium, blood flow phenomena, biophysics, etc. (see [20, 21, 27, 28, 29, 30, 35]). For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs [1, 2, 3, 21, 29]. Presently, in the literature several approaches/definitions of fractional integrals and derivatives are available, such as Riemann–Liouville definition, Caputo definition, Caputo–Hadamard, Hilfer. However, there is another kind of fractional derivatives that appears in the literature was introduced by Almeida which he called ψ -Caputo fractional operator [9]. Moreover, we can also

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highlight the papers [5, 10, 11, 31], in which many researchers turned to the existence and uniqueness of solutions for ψ -Caputo differential equations subject to various boundary conditions while some interesting details about initial and boundary value problems involving different kinds of fractional derivatives can be found in [4, 6, 7, 17, 22, 26]. In addition, in most of the existed articles, Schauder's, Krasnoselskii's, Darbo's, or Mönch's fixed point theorems have been employed to obtain the solutions of nonlinear fractional differential equations under some restrictive conditions (see [4, 8, 14] and references therein). On the other hand, in 2005 Sun and Zhang [34] generalized the definition of condensing operator to convex-power condensing operator. And based on the definition of this new kind of operator, they established a new fixed point theorem with respect to the convex-power condensing operator which generalizes the famous Sadovskii's fixed point theorem. After that, the fixed point theorem for the convex-power condensing operator due to Sun and Zhang was used by many researchers to study the existence of solutions for certain classes of nonlinear differential equations see, for instance, [16, 18, 23, 24, 25, 33, 36].

To the best of our knowledge, there are no papers dealing with the existence of solutions to a Cauchy problem with ψ -Caputo fractional derivative in Banach spaces. This gap is covered by the present paper. More specifically in this paper, we study the existence of solutions for certain classes of nonlinear differential equations involving the ψ -Caputo fractional-order in Banach spaces of the form:

$${}^c D_{a+}^{\alpha;\psi} u(t) = f(t, u(t)), \quad t \in J := [a, b], \quad \text{with } u(a) = u_a, \quad (1)$$

where ${}^c D_{a+}^{\alpha;\psi}$ is the ψ -Caputo fractional derivative of order α , $0 < \alpha \leq 1$ to be defined later, $f : J \times E \rightarrow E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$, and $u_a \in E$.

Our results are not only new in the given configuration but also correspond to some new situations associated with the specific values of the parameters involved in the given problem. For example, If we take $\psi(t) = t$, then the IVP (1) corresponds to the Cauchy problem containing the Caputo derivative, given by

$${}^c D_{a+}^{\alpha} u(t) = f(t, u(t)), \quad t \in J, \quad \text{with } u(a) = u_a.$$

Also noting that (1) is reduced to the Cauchy problem involving the Caputo–Hadamard derivative when $\psi(t) = \ln t$:

$${}^{CH} D_{a+}^{\alpha} u(t) = f(t, u(t)), \quad t \in J, \quad \text{with } u(a) = u_a.$$

Because of the essential differential between infinite-dimensional space and finite-dimensional space, the existence of solutions of fractional differential equations is no longer valid in abstract Banach space. The measure of noncompactness comes handy in such situations. The arguments are based on a new fixed point theorem with respect to the convex-power condensing operator combined with the technique of measures of noncompactness to establish the existence of solution for the IVP (1). As we generalized the space from the scalar space to the abstract space, our work includes the results of [4, 8, 17, 26].

The rest of this paper is organized as follows. In Section 2, we collect and derive some basic definitions, lemmas, and properties. In Section 3, we present the existence of solutions to the IVP problem (1). In Section 4, examples are given to illustrate our main results and the last section concludes this paper.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $\psi : J \rightarrow \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$, for all $t \in J$, and let $C(J, E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_{\infty} = \sup\{\|u(t)\|, t \in J\}.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \rightarrow E$, with the norm

$$\|u\|_1 = \int_a^b \|u(t)\| dt.$$

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2.1 ([13]). *The Kuratowski measure of noncompactness $\kappa(\cdot)$ defined on bounded set S of Banach space E is*

$$\kappa(S) := \inf\{\varepsilon > 0 : S = \cup_{k=1}^n S_k \text{ and } \text{diam}(S_k) \leq \varepsilon \text{ for } k = 1, 2, \dots, n\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.1 ([13]). *Let E be a Banach space and $A, B \subset E$ be bounded. The following properties are satisfied:*

- (1) $\kappa(A) \leq \kappa(B)$ if $A \subset B$;
- (2) $\kappa(A) = \kappa(\bar{A}) = \kappa(\text{conv } A)$, where $\text{conv } A$ means the convex hull of A ;
- (3) $\kappa(A) = 0$ if and only if \bar{A} is compact, where \bar{A} means the closure hull of A ;
- (4) $\kappa(\lambda A) = |\lambda| \kappa(A)$, where $\lambda \in \mathbb{R}$;
- (5) $\kappa(A \cup B) = \max\{\kappa(A), \kappa(B)\}$;
- (6) $\kappa(A + B) \leq \kappa(A) + \kappa(B)$, where $A + B = \{w \mid w = a + b, a \in A, b \in B\}$;
- (7) $\kappa(A + x) = \kappa(A)$, for any $x \in E$.

The following lemmas are needed in our argument.

Lemma 2.2 ([12]). *Let E be a Banach space, and let $B \subset C(J, E)$ be bounded and equicontinuous. Then $\kappa(B(\cdot))$ is continuous on J and $\kappa_C(B) = \max_{t \in J} \kappa(B(t))$.*

Lemma 2.3 ([15]). *Let E be a Banach space and let $B \subset E$ be bounded. Then for each ε , there is a sequence $\{u_n\}_{n=1}^\infty \subset B$, such that*

$$\kappa(B) \leq 2\kappa(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

We call $B \subset L^1(J, E)$ uniformly integrable if there exists $\eta \in L^1(J, \mathbb{R}^+)$ such that

$$\|u(s)\| \leq \eta(s), \text{ for all } u \in B \text{ and a.e. } s \in J.$$

Lemma 2.4 ([19]). *If $\{u_n\}_{n=1}^\infty \subset L^1(J, E)$ is uniformly integrable, then $t \mapsto \kappa(\{u_n(t)\}_{n=1}^\infty)$ is measurable, and*

$$\kappa\left(\left\{\int_a^t u_n(s) ds\right\}_{n=1}^\infty\right) \leq 2 \int_a^t \kappa(\{u_n(s)\}_{n=1}^\infty) ds.$$

The following fixed point theorem with respect to convex-power condensing operator which was introduced by Sun and Zhang [34] plays a key role in the proof of our main results, see also [24, 25] for more applications of this theorem.

Definition 2.2. *Let X be a real Banach space. If $\mathcal{A} : X \rightarrow X$ is a continuous and bounded operator, there exist $u_0 \in X$ and a positive integer n_0 such that for any bounded and nonprecompact subset $S \subset X$,*

$$\kappa(\mathcal{A}^{(n_0, u_0)}(S)) < \kappa(S), \tag{2}$$

where

$$\mathcal{A}^{(1, u_0)}(S) \equiv \mathcal{A}(S), \quad \mathcal{A}^{(n, u_0)}(S) = \mathcal{A}(\overline{\text{co}}\{\mathcal{A}^{(n-1, u_0)}(S)\}), \quad n = 2, 3, \dots$$

Then we call \mathcal{A} a convex-power condensing operator about u_0 and n_0 .

Lemma 2.5. (Fixed point theorem with respect to convex-power condensing operator, [34]) Let X be a real Banach space, and let $B \subset X$ be a bounded, closed and convex set in X . If there exist $u_0 \in B$ and a positive integer n_0 such that $\mathcal{A} : B \rightarrow B$ be a convex-power condensing operator about u_0 and n_0 , then the operator \mathcal{A} has at least one fixed point in B .

Remark 2.1. If $n_0 = 1$ in (2), then fixed point theorem with respect to convex-power condensing operator (see Lemma 2.5) will degrade into the famous Sadovskii’s fixed point theorem. Noticed that Lemma 2.5 requires the operator \mathcal{A} is neither condensing nor completely continuous. Therefore, fixed point theorem with respect to convex-power condensing operator is the generalization of the well-known Sadovskii’s fixed point theorem.

Definition 2.3 ([37]). A function $f : J \times E \rightarrow E$ is said to satisfy the Carathéodory conditions, if the following hold

- $f(t, u)$ is measurable with respect to t for $u \in E$,
- $f(t, u)$ is continuous with respect to $u \in E$ a.e. $t \in J$.

Now, we give some results and properties from the theory of fractional calculus. We begin by defining ψ -Riemann–Liouville fractional integrals and derivatives. In what follows,

Definition 2.4 ([9, 21]). For $\alpha > 0$, the left-sided ψ -Riemann–Liouville fractional integral of order α for an integrable function $u : J \rightarrow \mathbb{R}$ with respect to another function $\psi : J \rightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in J$ is defined as follows

$$I_{a^+}^{\alpha;\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) ds, \tag{3}$$

where Γ is the gamma function. Note that Eq. (3) is reduced to the Riemann–Liouville and Hadamard fractional integrals when $\psi(t) = t$ and $\psi(t) = \ln t$, respectively.

Definition 2.5 ([9]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(J)$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ -Riemann–Liouville fractional derivative of a function u of order α is defined by

$$D_{a^+}^{\alpha;\psi} u(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{n-\alpha;\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$.

Definition 2.6 ([9]). Let $n \in \mathbb{N}$ and let $\psi, u \in C^n(J)$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ -Caputo fractional derivative of u of order α is defined by

$${}^c D_{a^+}^{\alpha;\psi} u(t) = I_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we will use the abbreviated symbol $u_{\psi}^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n u(t)$.

From the definition, it is clear that,

$${}^c D_{a^+}^{\alpha;\psi} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} u_{\psi}^{[n]}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ u_{\psi}^{[n]}(t), & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{4}$$

This generalization (4) yields the Caputo fractional derivative operator when $\psi(t) = t$. Moreover, for $\psi(t) = \ln t$, it gives the Caputo–Hadamard fractional derivative.

We note that if $u \in C^n(J)$ the ψ -Caputo fractional derivative of order α of u is determined as

$${}^c D_{a^+}^{\alpha;\psi} u(t) = D_{a^+}^{\alpha;\psi} \left[u(t) - \sum_{k=0}^{n-1} \frac{u_{\psi}^{[k]}(a)}{k!} (\psi(t) - \psi(a))^k \right].$$

(See, for instance, [9, Theorem 3]).

Lemma 2.6 ([10]). *Let $\alpha, \beta > 0$, and $u \in L^1(J)$. Then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} u(t) = I_{a^+}^{\alpha+\beta;\psi} u(t)$, a.e. $t \in J$. In particular, If $u \in C(J)$. Then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} u(t) = I_{a^+}^{\alpha+\beta;\psi} u(t)$, $t \in J$.*

Next, we recall the property describing the composition rules for fractional ψ -integrals and ψ -derivatives.

Lemma 2.7 ([10]). *Let $\alpha > 0$, The following holds:*

If $u \in C[a, b]$ then ${}^c D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} u(t) = u(t)$, $t \in [a, b]$.

If $u \in C^n(J)$, $n - 1 < \alpha < n$. Then $I_{a^+}^{\alpha;\psi} {}^c D_{a^+}^{\alpha;\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_{\psi}^{[k]}(a)}{k!} [\psi(t) - \psi(a)]^k$, for all $t \in [a, b]$. In particular, if $0 < \alpha < 1$, we have $I_{a^+}^{\alpha;\psi} {}^c D_{a^+}^{\alpha;\psi} u(t) = u(t) - u(a)$.

Lemma 2.8 ([10, 21]). *Let $t > a$, $\alpha \geq 0$, and $\beta > 0$. Then*

- $I_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(a))^{\beta+\alpha-1}$,
- ${}^c D_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(a))^{\beta-\alpha-1}$,
- ${}^c D_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^k = 0$, for all $k \in \{0, \dots, n - 1\}$, $n \in \mathbb{N}$.

Remark 2.2. *Note that for an abstract function $u : J \rightarrow E$, the integrals which appear in the previous definitions are taken in Bochner’s sense (see, for instance, [32]).*

3. Main Results

Let us recall the definition and lemma of a solution for problem (1). First of all, we define what we mean by a solution for the IVP (1).

Definition 3.1. *A function $u \in C(J, E)$ is said to be a solution of Eq. (1) if u satisfies the equation ${}^c D_{a^+}^{\alpha;\psi} u(t) = f(t, u(t))$, a.e. on J , and the condition $u(a) = u_a$.*

For the existence of solutions for the problem (1) we need the following lemma:

Lemma 3.1. *For a given $h \in L^1(J, \mathbb{R})$, the unique solution of the linear fractional initial value problem*

$${}^c D_{a^+}^{\alpha;\psi} u(t) = h(t), \quad 0 < \alpha \leq 1, \quad t \in J, \quad \text{with } u(a) = u_a, \tag{5}$$

is given by

$$u(t) = u_a + I_{a^+}^{\alpha;\psi} h(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} h(s) ds. \tag{6}$$

Proof. Taking the ψ -Riemann–Liouville fractional integral of order α to the first equation of (5), we get

$$u(t) = I_{a^+}^{\alpha;\psi} h(t) + k_0, \quad k_0 \in \mathbb{R}. \tag{7}$$

Substituting $t = a$ in (7) and using the boundary condition of (5), it yields

$$k_0 = u_a.$$

Substituting the values of k_0 into (7), we get the integral equation (6). The converse follows by direct computation which completes the proof. \square

As a result of Lemma 3.1, the IVP (1) can be converted to an integral equation which takes the following form

$$u(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds. \quad (8)$$

Now, we shall present our main result concerning the existence of solutions of problem (1). Let us introduce the following hypotheses

(H1) The function $f : J \times E \rightarrow E$ satisfies Carathéodory conditions.

(H2) There exist $p_f \in L^\infty(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|f(t, u)\| \leq p_f(t)\phi(\|u\|), \quad \text{for a.e. } t \in J, \text{ and all } u \in E.$$

(H3) There exists a positive constant L , such that for any bounded and countable set $D \subset E$ and a.e. $t \in J$,

$$\kappa(f(t, D)) \leq L\kappa(D).$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1).

Theorem 3.1. *Assume that the hypotheses (H1)–(H3) are satisfied and there exists a constant $R > 0$ such that*

$$R \geq \frac{\|p_f\|_{L^\infty} (\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \phi(R) + \|u_a\|, \quad (9)$$

then the problem (1) has at least one solution defined on J .

Proof. Consider the operator $\mathcal{N} : C(J, E) \rightarrow C(J, E)$ defined by:

$$\mathcal{N}u(t) = u_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds. \quad (10)$$

It is obvious that \mathcal{N} is well defined due to (H1) and (H2). Then, the fractional integral equation (8) can be written as the following operator equation

$$u = \mathcal{N}u. \quad (11)$$

Thus, the existence of a solution for Eq. (1) is equivalent to the existence of a fixed point for operator \mathcal{N} which satisfies operator equation (11).

Define the set

$$B_R = \{w \in C(J, E) : \|w\|_\infty \leq R\}.$$

Notice that B_R is closed, convex and bounded subset of the Banach space $C(J, E)$. We shall show that the operator \mathcal{N} satisfies all the assumptions of Lemma 2.5. We split the proof into four steps.

Step 1: The operator \mathcal{N} maps the set B_R into itself. By the assumption (H2), we have

$$\begin{aligned} \|\mathcal{N}u(t)\| &\leq \|u_a\| + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|f(s, u(s))\| ds \\ &\leq \|u_a\| + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} p_f(s)\phi(\|u(s)\|) ds \\ &\leq \|u_a\| + \frac{\|p_f\|_{L^\infty} \phi(\|u\|)}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ &\leq \|u_a\| + \frac{\|p_f\|_{L^\infty} \phi(\|u\|) (\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \leq R. \end{aligned}$$

Thus

$$\|\mathcal{N}u\| \leq R.$$

This proves that \mathcal{N} transforms the ball B_R into itself.

Step 2: The operator \mathcal{N} is continuous. Suppose that $\{u_n\}$ is a sequence such that $u_n \rightarrow u$ in B_R as $n \rightarrow \infty$. It is easy to see that $f(s, u_n(s)) \rightarrow f(s, u(s))$, as $n \rightarrow +\infty$, due to the Carathéodory continuity of f . On the other hand taking (H2) into consideration we get the following inequality:

$$\psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|f(s, u_n(s)) - f(s, u(s))\| \leq 2\phi(R)\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}.$$

We notice that since the function $s \mapsto 2\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\phi(R)$ is Lebesgue integrable over $[a, t]$. This fact together together with the Lebesgue dominated convergence theorem imply that

$$\frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|f(s, u_n(s)) - f(s, u(s))\| ds \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

It follows that $\|\mathcal{N}u_n - \mathcal{N}u\| \rightarrow 0$ as $n \rightarrow +\infty$, which implies the continuity of the operator \mathcal{N} .

Step 3: $\mathcal{N}(B_R)$ is equicontinuous. For any $a < t_1 < t_2 < b$ and $u \in B_R$, we get

$$\begin{aligned} & \|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \psi'(s) [(\psi(t_1) - \psi(s))^{\alpha-1} - (\psi(t_2) - \psi(s))^{\alpha-1}] \|f(s, u(s))\| ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} \|f(s, u(s))\| ds \\ & \leq \frac{\|p_f\|_{L^\infty} \phi(R)}{\Gamma(\alpha + 1)} [(\psi(t_1) - \psi(a))^\alpha + 2(\psi(t_2) - \psi(t_1))^\alpha - (\psi(t_2) - \psi(a))^\alpha] \\ & \leq \frac{2\|p_f\|_{L^\infty} \phi(R)}{\Gamma(\alpha + 1)} (\psi(t_2) - \psi(t_1))^\alpha, \end{aligned}$$

where we have used the fact that $(\psi(t_1) - \psi(a))^\alpha - (\psi(t_2) - \psi(a))^\alpha \leq 0$. Therefore,

$$\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \leq \frac{2\|p_f\|_{L^\infty} \phi(R)}{\Gamma(\alpha + 1)} (\psi(t_2) - \psi(t_1))^\alpha.$$

As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero independently of $u \in B_R$. Hence, we conclude that $\mathcal{N}(B_R) \subseteq C(J, E)$ is bounded and equicontinuous.

Step 4: Next, we prove that $\mathcal{N} : Q \rightarrow Q$ is a convex-power condensing operator, where $Q = \overline{\text{co}}\mathcal{N}(B_R)$ and $\overline{\text{co}}$ means the closure of convex hull. Then one can easily verify that the operator \mathcal{N} maps Q into itself and $Q \subset C(J, E)$ is equicontinuous. Let $u_0 \in Q$. In the following, we will prove that there exists a positive integer n_0 such that for any bounded and nonprecompact subset $B \subset Q$

$$\kappa_C \left(\mathcal{N}^{(n_0, u_0)}(B) \right) < \kappa_C(B). \tag{12}$$

For any $B \subset Q$ and $u_0 \in Q$, by the definition of operator $\mathcal{N}^{(n, u_0)}$ and the equicontinuity of Q , we get that $\mathcal{N}^{(n, u_0)}(B) \subset B_R$ is also equicontinuous. Therefore, we know from Lemma 2.2 that

$$\kappa_C \left(\mathcal{N}^{(n, u_0)}(B) \right) = \max_{t \in J} \kappa \left(\mathcal{N}^{(n, u_0)}(B)(t) \right), \quad n = 1, 2, \dots \tag{13}$$

Let $\varepsilon > 0$. By Lemma 2.3, there exist sequences $\{u_k\}_{k=1}^\infty \subset B$ such that

$$\begin{aligned} & \kappa \left(\mathcal{N}^{(1, u_0)}(B)(t) \right) = \kappa \left(\mathcal{N}(B)(t) \right) \leq \\ & 2\kappa \left\{ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \{u_k(s)\}_{k=1}^\infty) ds \right\} + \varepsilon. \end{aligned}$$

Next, by Lemma 2.4 and the properties of κ and (H3) we have

$$\begin{aligned} \kappa \left(\mathcal{N}^{(1,u_0)}(B)(t) \right) &\leq \\ &4 \left\{ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \kappa(f(s, \{u_k(s)\}_{k=1}^\infty)) ds \right\} + \varepsilon \\ &\leq 4L \left\{ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \kappa(\{u_k(s)\}_{k=1}^\infty) ds \right\} + \varepsilon \\ &\leq \frac{4L\kappa(B)}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds + \varepsilon \\ &\leq \frac{4L(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \kappa(B) + \varepsilon. \end{aligned}$$

As the last inequality is true, for every $\varepsilon > 0$, we infer

$$\kappa \left(\mathcal{N}^{(1,u_0)}(B)(t) \right) \leq \frac{4L(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} \kappa(B). \tag{14}$$

Using Lemma 2.3 one more time, we see that for any $\varepsilon > 0$, there is a sequence $\{v_k\}_{k=1}^\infty \subset \overline{c\mathcal{O}} \{ \mathcal{N}^{(1,u_0)}(B), u_0 \}$, such that

$$\begin{aligned} \kappa(\mathcal{N}^{(2,u_0)}(B)(t)) &= \kappa \left(\mathcal{N}(\overline{c\mathcal{O}} \{ \mathcal{N}^{(1,u_0)}(B), u_0 \})(t) \right) \\ &\leq 2\kappa \left\{ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \{v_k(s)\}_{k=1}^\infty) ds \right\} + \varepsilon \\ &\leq \frac{4L}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \kappa(\{v_k(s)\}_{k=1}^\infty) ds + \varepsilon \\ &\leq \frac{4L}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \kappa \left(\overline{c\mathcal{O}} \{ \mathcal{N}^{(1,u_0)}(B), u_0 \}(s) \right) ds + \varepsilon \\ &\leq \frac{4L}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \kappa \left(\{ \mathcal{N}^{(1,u_0)}(B) \}(s) \right) ds + \varepsilon \\ &\leq \frac{(4L)^2}{\Gamma(\alpha + 1)} \kappa(B) \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} (\psi(s) - \psi(a))^\alpha ds + \varepsilon. \end{aligned}$$

Also note that

$$\begin{aligned} &\int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1} (\psi(s) - \psi(a))^\alpha}{\Gamma(\alpha)} ds = \\ &\frac{(\psi(t) - \psi(a))^{2\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - y)^{\alpha-1} y^\alpha dy \\ &= \frac{(\psi(t) - \psi(a))^{2\alpha}}{\Gamma(\alpha)} B(\alpha, \alpha + 1) \\ &= \frac{(\psi(t) - \psi(a))^{2\alpha}}{\Gamma(2\alpha + 1)} \Gamma(\alpha + 1), \end{aligned}$$

where we have used the variable substitution $y = \frac{\psi(s) - \psi(a)}{\psi(t) - \psi(a)}$. Using the above arguments, we get

$$\kappa(\mathcal{N}^{(2,u_0)}(B)(t)) \leq \frac{(4L)^2 (\psi(t) - \psi(a))^{2\alpha}}{\Gamma(2\alpha + 1)} \kappa(B) + \varepsilon.$$

As the last inequality is true, for every $\varepsilon > 0$, we infer

$$\kappa \left(\mathcal{N}^{(2,u_0)}(B)(t) \right) \leq \frac{(4L)^2 (\psi(t) - \psi(a))^{2\alpha}}{\Gamma(2\alpha + 1)} \kappa(B).$$

It can be shown by mathematical induction that

$$\kappa \left(\mathcal{N}^{(n,u_0)}(B)(t) \right) \leq \frac{(4L)^n (\psi(t) - \psi(a))^{n\alpha}}{\Gamma(n\alpha + 1)} \kappa(B). \quad (15)$$

Hence, by (13) and (15), we get that

$$\kappa_C \left(\mathcal{N}^{(n,u_0)}(B) \right) = \max_{t \in I} \kappa \left(\mathcal{N}^{(n,u_0)}(B)(t) \right) \leq \frac{(4L)^n (\psi(b) - \psi(a))^{n\alpha}}{\Gamma(n\alpha + 1)} \kappa(B). \quad (16)$$

By using the well-known Stirling formula, we know there exists a constant $0 < \theta < 1$, such that

$$\Gamma(1 + n\alpha) = \sqrt{2\pi n\alpha} \left(\frac{n\alpha}{e} \right)^{n\alpha} e^{\frac{\theta}{12n\alpha}}, \quad \alpha > 0.$$

It follows that

$$\lim_{n \rightarrow +\infty} \frac{(4L)^n \cdot (\psi(b) - \psi(a))^{n\alpha}}{\Gamma(1 + n\alpha)} = \lim_{n \rightarrow +\infty} \frac{(4L)^n \cdot (\psi(b) - \psi(a))^{n\alpha}}{\sqrt{2\pi n\alpha} \left(\frac{n\alpha}{e} \right)^{n\alpha} e^{\frac{\theta}{12n\alpha}}} = 0.$$

Therefore, there must exist a positive integer n_0 , which is large enough, such that

$$\frac{(4L)^{n_0} \cdot (\psi(b) - \psi(a))^{n_0\alpha}}{\Gamma(1 + n_0\alpha)} < 1. \quad (17)$$

Hence, from (16) and (17) we know that (12) is satisfied, which means that $\mathcal{N} : Q \rightarrow Q$ is a convex-power condensing operator. It follows from Lemma 2.5 that the operator \mathcal{N} defined by (10) has at least one fixed point $u \in Q$, which is just the solution of initial value problem (1). This completes the proof Theorem 3.1. \square

Remark 3.1. From (13) and (14) of Theorem 3.1 we know that if we assume

$$\frac{4L (\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha + 1)} < 1, \quad (18)$$

directly, we can apply the famous Sadovskii's fixed point theorem to obtain the results of Theorem 3.1. However, from the above arguments, one can find that we do not need this redundant condition (18) by virtue of fixed point theorem with respect to convex-power condensing operator.

4. Examples

In this section we give two examples to illustrate the usefulness of our main result. Let

$$E = c_0 = \{u = (u_1, u_2, \dots, u_n, \dots) : u_n \rightarrow 0 (n \rightarrow \infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|u\|_\infty = \sup_{n \geq 1} |u_n|.$$

Example 4.1. Consider the following initial value problem of a fractional differential posed in c_0 :

$$\begin{cases} {}^{CH}D_{1+}^{\frac{1}{2}}u(t) = f(t, u(t)), & t \in J := [1, e], \\ u(1) = (0, 0, \dots, 0, \dots). \end{cases} \quad (19)$$

Note that, this problem is a particular case of IVP (1), where

$$\alpha = \frac{1}{2}, a = 1, b = e, \psi(t) = \ln(t),$$

and $f : J \times c_0 \rightarrow c_0$ given by

$$f(t, u) = \left\{ \frac{1}{(t-1)^2 + 2} \left(\frac{1}{n^2} + \arctan(|u_n|) \right) \right\}_{n \geq 1},$$

for $t \in J, u = \{u_n\}_{n \geq 1} \in c_0$.

It is clear that condition (H1) holds, and as

$$\begin{aligned} \|f(t, u)\| &\leq \frac{1}{(t-1)^2 + 2} (1 + \|u\|) \\ &= p_f(t)\phi(\|u\|). \end{aligned}$$

Therefore, assumption (H2) of Theorem 3.1 is satisfied with

$$p_f(t) = \frac{1}{(t-1)^2 + 2}, \quad t \in J,$$

and

$$\phi(x) = 1 + x, \quad x \in [0, \infty).$$

On the other hand, for any bounded set $D \subset c_0$, we have

$$\kappa(f(t, D)) \leq \frac{1}{2}\kappa(D), \quad \text{a.e. } t \in J.$$

Hence (H3) is satisfied. Now, we will prove that the inequality (9) is satisfied. Since $\phi(x) = 1 + x$, then we have to find $R > 0$ such that

$$(1 + R) \frac{1}{2\Gamma(\frac{3}{2})} \leq R,$$

thus

$$R \geq \frac{1}{\sqrt{\pi} - 1} = 1.2946.$$

Then R can be chosen as $R = 1.5 \geq 1.2946$. Consequently, all hypotheses of Theorem 3.1 are satisfied and we conclude that the IVP (19) has at least one solution $u \in C(J, c_0)$.

Let now

$$E = \ell^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots), \sum_{n=1}^{\infty} |u_n| < \infty \right\},$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Example 4.2. Consider the following initial value problem of a fractional differential posed in ℓ^1

$$\begin{cases} {}^c D_{0+}^{\frac{1}{2}} u(t) = f(t, u(t)), t \in J := [0, 1], \\ u(0) = \left(\frac{1}{\sqrt{\pi}}, 0, \dots, 0, \dots \right). \end{cases} \quad (20)$$

Note that this problem is a particular case of BVP (1), where $\alpha = \frac{1}{2}$, $a = 0$, $b = 1$, $\psi(t) = t$, and $f : J \times \ell^1 \rightarrow \ell^1$ given by $f(t, u) = \left\{ \frac{1}{t^2+2} \left(\frac{t}{2^n} + \frac{u_n}{\|u\|+1} \right) \right\}_{n \geq 1}$, for $t \in J$, $u = \{u_n\}_{n \geq 1} \in \ell^1$. It is clear that condition (H1) hold, and as

$$\|f(t, u)\| \leq \frac{1}{t^2+2} (1 + \|u\|) = p_f(t) \phi(\|u\|).$$

Therefore, assumption (H2) of Theorem 3.1 is satisfied with

$$p_f(t) = \frac{1}{t^2+2}, t \in J, \text{ and } \phi(x) = 1 + x, x \in [0, \infty).$$

On the other hand, for any bounded set $D \subset c_0$, we have

$$\kappa(f(t, D)) \leq \frac{1}{2} \kappa(D), \text{ a.e. } t \in J.$$

Hence (H3) is satisfied. Now, we will prove that the inequality (9) is satisfied. Since $\phi(x) = 1 + x$, then we have to find $R > 0$ such that

$$\frac{1}{\sqrt{\pi}} + (1 + R) \frac{1}{2\Gamma\left(\frac{3}{2}\right)} \leq R,$$

thus

$$R \geq \frac{2}{\sqrt{\pi} - 1} = 2.5892.$$

Then R can be chosen as $R = 3 \geq 2.5892$. Consequently, all the hypothesis of Theorem 3.1 are satisfied and we conclude that the IVP (20) has at least one solution $u \in C(J, \ell^1)$.

5. Conclusion

We discussed the existence of solutions for a Cauchy problem with ψ -Caputo fractional derivative that is reduced to the well known fractional derivatives like Caputo, Caputo–Hadamard for some particular cases of $\psi(t)$. The proofs are based essentially on a new fixed point theorem with respect to convex-power condensing operator combined with the technique of measure of noncompactness. Also, we provided two examples to illustrate our results.

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