Sakarya University Journal of Science
ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University |
http://www.saujs.sakarya.edu.tr/

Title: Hurwitz Stability of Matrix Segment and The Common Solution Set of 2 and 3-Dimensional Lyapunov Equations

Authors: Şerife Yılmaz
Recieved: 2019-06-03 10:47:26

Accepted: 2020-01-16 15:43:22

Article Type: Research Article
Volume: 24
Issue: 2
Month: April
Year: 2020
Pages: 357-364

How to cite
Şerife Yılmaz; (2020), Hurwitz Stability of Matrix Segment and The Common Solution Set of 2 and 3-Dimensional Lyapunov Equations. Sakarya University Journal of Science, 24(2), 357-364, DOI: 10.16984/saufenbilder.573551
Access link
http://www.saujs.sakarya.edu.tr/tr/issue/52471/573551

Sakarya University Journal of Science 24(2), 357-364, 2020

Hurwitz Stability of Matrix Segment and The Common Solution Set of 2 and 3-Dimensional Lyapunov Equations

Şerife YILMAZ[1]

Abstract

In this study, a necessary and sufficient condition is given for the stability of the convex combinations of n-dimensional two Hurwitz stable matrices. There is a close relationship between Hurwitz stability of the matrix segment and common solution to the Lyapunov equations corresponding to those matrices. Therefore, the results obtained in this area are important. In the case of existence, an algorithm that determines common solutions set is also given. A number of illustrative examples using this algorithm are given.

Keywords: Hurwitz stability, matrix segment, common quadratic Lyapunov function

1. INTRODUCTION

Let A_1 and A_2 be n-dimensional square real matrices, that is $A_1, A_2 \in \mathbb{R}^{n\times n}$. If all eigenvalues of a square matrix lie in the open left half plane, it is called Hurwitz stable matrix. A stable matrix can also be characterized by Lyapunov inequality: if A_1 is a Hurwitz stable matrices, there exists a positive definite P such that

$$A_1^T P + PA_1 < 0 \tag{1}$$

is hold (see [1,2]). When considering stable matrices A_1 and A_2 if the following inequalities

$$\begin{aligned} A_1^T P + PA_1 < 0, \\ A_2^T P + PA_2 < 0 \end{aligned} \tag{2}$$

are simultaneously satisfied for a $P > 0$, the matrix P is called common solution to the matrices A_1 and A_2.

The problem of the existence of a common $P > 0$ has been extensively investigated for the last two decades (see [3-8], and references therein). A

[1]Department of Mathematics and Sciences Education, Faculty of Education, Burdur Mehmet Akif Ersoy University, Istiklal Campus 15030 Burdur, Turkey. E-mail: serifeyilmaz@mehmetakif.edu.tr, Orcid Id: 0000-0002-7561-3288

sufficient condition for the asympotitical stability of the linear system given by the finite number of matrices, which are called switched system, is the existence of their common solution. With the exception of some special cases (for instance, second order matrices), the theoretical solution to the general n-dimensional problem has not been found yet [4-8].

Define the matrix segment

$$[A_1, A_2] = \{C(\alpha): C(\alpha) = \alpha A_1 + (1-\alpha)A_2, \alpha \in [0,1]\}. \quad (3)$$

With the stability of the segment $[A_1, A_2]$, we mean that all matrix $A \in [A_1, A_2]$ is stable.

In [5], a necessary and sufficient condition for the existence of a common solution to second order matrices is given. According to this result, the stability of the matrix segments $[A_1, A_2]$ and $[A_1, A_2^{-1}]$ are equivalent to the existence of a common solution $P > 0$.

Notice that the existence of the solution of the problem is related to the stability of the convex combinations of the matrices.

The paper is organized as follows: Section 2 introduces the relation between bialternate product and stability of matrices. A necessary and sufficient condition for the segment stability will be derived via this product. Section 3 considers the common solution to the Lyapunov equation for two and three-dimensional matrices. Section 4 considers finding common solutions. By using the minimization of the functions defined on a box, an algorithm will be constructed. In the case of existence, the bisection method determinates common solutions. A number of illustrative examples are provided.

2. THE STABILITY CRITERIA FOR CONVEX COMBINATIONS OF TWO MATRICES

For 2×2 real matrices, a necessary and sufficient condition for the matrix segment in (3) to be stable is that both A_1 and A_2 are stable, and that the matrix $A_1 A_2^{-1}$ has no negative eigenvalues (see [5,9,10]).

Let A be an $n \times n$ matrix. We denote the bialternate product of $2A$ and the identity matrix I_n by $L(A)$:

$$L(A) = (2A) \cdot I_n.$$

As usual the bialternate product is denoted by "·". The dimension of $L(A)$ is $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ and if the eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$ then the eigenvalues of $L(A)$ are written $\lambda_i + \lambda_j$ where $i = 1,2,\dots,n-1$ and $j = i+1, i+2, \dots, i+n$ (see [11-13]).

Now we give the following stability theorem for a matrix A, in terms of the above mentioned bialternate product.

Theorem 1 [14, p. 37]: The matrix A is stable if and only if the coefficients of the characteristic polynomials of the matrices A and $L(A)$

$$\begin{aligned} p_A(s) &= \det(sI_n - A) \\ &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0, \\ P_{L(A)}(s) &= \det\big(sI_m - L(A)\big) \\ &= s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0. \end{aligned}$$

are positive. That is, $a_i > 0$ and $b_j > 0$ for $i = 1,2,\dots,n-1$, $j = 1,2,\dots m-1$, where $m = n(n-1)/2$.

Note that in the case of A is stable, the following inequalities hold

$$\begin{aligned} p_A(0) &= \det(0 \cdot I_n - A) \\ &= (-1)^n \det(A) > 0, \\ P_{L(A)}(0) &= \det\big(0 \cdot I_m - L(A)\big) \\ &= (-1)^m \det\big(L(A)\big) > 0. \end{aligned}$$

Lemma 1: Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be stable matrices. For all $\alpha \in [0,1]$, $(-1)^n \det C(\alpha) > 0$ if and only if the matrix $A_1 A_2^{-1}$ has no negative real eigenvalues.

Proof: Since A_1 and A_2 are stable, $(-1)^n \det(A_1) > 0$ and $(-1)^n \det(A_2) > 0$.

Sufficiency: For $\alpha \in (0,1)$,

$$\begin{aligned}\det C(\alpha) &= \det[\alpha A_1 + (1-\alpha)A_2] \\ &= \det([\alpha A_1 A_2^{-1} + (1-\alpha)I_n]A_2) \\ &= \det[\alpha A_1 A_2^{-1} + (1-\alpha)I_n]\det(A_2) \\ &= \alpha^n \det\left[A_1 A_2^{-1} + \frac{1-\alpha}{\alpha} I_n\right]\det(A_2).\end{aligned}$$

Since $\frac{1-\alpha}{\alpha} \in (0,\infty)$ and $A_1 A_2^{-1}$ has no negative real eigenvalues,

$$\det\left[A_1 A_2^{-1} + \frac{1-\alpha}{\alpha} I_n\right] \neq 0.$$

Hence, $\det C(\alpha) > 0$ for $\alpha \in (0,1)$.

Necessity: If $(-1)^n \det C(\alpha) > 0$ for all $\alpha \in (0,1)$ then $\det\left[A_1 A_2^{-1} + \frac{1-\alpha}{\alpha} I_n\right] \neq 0$ and so the matrix $A_1 A_2^{-1}$ has no negative eigenvalues.

Lemma 2: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable matrices and $C(\alpha)$ be as in (3). For all $\alpha \in [0,1]$,

$$(-1)^m \det L(C(\alpha)) > 0$$

if and only if the matrix $L(A_1)L^{-1}(A_2)$ has no negative real eigenvalues.

Proof is analogously to Lemma 1.

Theorem 2: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable matrices and $C(\alpha)$ be as in (3). For all $\alpha \in [0,1]$, $C(\alpha)$ is stable if and only if the matrices $A_1 A_2^{-1}$ and $L(A_1)L^{-1}(A_2)$ have no negative real eigenvalues.

Proof: From Theorem 1, $(-1)^n \det C(\alpha) > 0$ and $(-1)^m \det L(C(\alpha)) > 0$ for all $\alpha \in [0,1]$ since $C(\alpha)$ is stable. Therefore, the matrices $A_1 A_2^{-1}$ and $L(A_1)L^{-1}(A_2)$ have no negative eigenvalues from Lemma 1 and 2.

Sufficiency: Assume that $A_1 A_2^{-1}$ and $L(A_1)L^{-1}(A_2)$ have no negative eigenvalues. Consider the eigenvalues of the matrix $C(\alpha)$. By the continuity theorem of eigenvalues [16, p. 52], there exists continuous functions $\lambda_i: [0,1] \to C$ $(i = 1,2,\dots,n)$ such that $\lambda_1(\alpha), \lambda_2(\alpha), \dots, \lambda_n(\alpha)$ are the eigenvalues of $C(\alpha)$. Here $Re\,\lambda_i(0) < 0$ $(i = 1,2,\dots,n)$, since the matrix $C(0)$ is stable. Proceeding by contraposition, suppose that $C(\alpha_*)$ is not Hurwitz stable for an $\alpha_* \in (0,1)$. Therefore, there exists an index $i_0 \in \{1,2,\dots,n\}$ such that $Re\,\lambda_{i_0}(\alpha_*) \geq 0$. In view of the continuity of $\lambda_{i_0}(\alpha)$ with respect to α, there must exists an $\tilde{\alpha} \in (0,1)$ such that $Re\,\lambda_{i_0}(\tilde{\alpha}) = 0$. The matrix $C(\tilde{\alpha})$ has an eigenvalue which lies on the imaginary axes. If the eigenvalue is real then

$$\det C(\tilde{\alpha}) = \lambda_1(\tilde{\alpha}).\cdots.\lambda_{i_0}(\tilde{\alpha}).\cdots.\lambda_n(\tilde{\alpha}) = 0$$

which contradicts the result of Lemma 1. If the eigenvalue is not real, $\lambda_{i_0}(\tilde{\alpha}) = j\tilde{\omega}$ where $\tilde{\omega} > 0$. The complex conjugate of $j\tilde{\omega}$ is also an eigenvalue of $C(\tilde{\alpha})$. The matrix $L(C(\tilde{\alpha}))$ has the eigenvalue $j\tilde{\omega} + (-j\tilde{\omega}) = 0$ and this imply $\det L(C(\tilde{\alpha})) = 0$ which contradicts Lemma 2. This contradictions show that $C(\alpha)$ is stable for all $\alpha \in [0,1]$.

3. THE COMMON SOLUTION TO THE LYAPUNOV EQUATION FOR TWO AND THREE-DIMENSIONAL MATRICES

In this section, we can give the important theorem on common quadratic solution two−dimensional Lyapunov equations for two stable matrices.

Theorem 3 [5]: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be stable matrices. A necessary and sufficient condition for the matrices A_1 and A_2 have common solution to its Lyapunov equation (2) is that the matrices $A_1 A_2$ and $A_1 A_2^{-1}$ have no negative real eigenvalue. An equivalent condition is that the segments $[A_1, A_2]$ and $[A_1, A_2^{-1}]$ are stable.

Note that, if A is a 2×2 dimensional matrix then the matrix $L(A)$ is a number and is equal to trace(A).

For the common solution set of two-dimensional Lyapunov equation of the matrices A_1 and A_2, define the symmetric matrices

$$P(x) = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, Q_2(y) = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix},$$

$$R_2(z) = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}.$$

Let A_1 and A_2 be stable matrices. Assume that A_1A_2 and $A_1A_2^{-1}$ have no negative real eigenvalue. Consider the following Lyapunov equations:

$$A_1^T P(x) + P(x)A_1 = -Q_2(y),$$
$$A_2^T P(x) + P(x)A_2 = -R_2(z)$$

where $x, y, z \in \mathbb{R}^3$. For given $Q_2(y) > 0$, there exists an $x \in \mathbb{R}^3$ such that the matrix $P(x) > 0$ is a unique solution to the first equation. That is, from the solution of the first equation

$$x = (\phi_1(y), \phi_2(y), \phi_3(y))$$

is obtained. Analogously, for $R_2(z) > 0$ there exists an $x \in \mathbb{R}^3$ such that the matrix $P(x) > 0$ is a unique solution to the second equation. As a result,

$$x = (\eta_1(z), \eta_2(z), \eta_3(z))$$

is obtained. Finally, if these two results are combined, the equation for the common solution is

$$(\phi_1(y), \phi_2(y), \phi_3(y)) = (\eta_1(z), \eta_2(z), \eta_3(z)).$$

From this linear equations system,

$$y = (\gamma_1(z), \gamma_2(z), \gamma_3(z)).$$

We investigate three-dimensional box where the symmetric matrix $R_2(z)$ is positive definite on it.

The matrix

$$Q_2(z) = \begin{bmatrix} \gamma_1(z) & \gamma_2(z) \\ \gamma_2(z) & \gamma_3(z) \end{bmatrix}$$

must be positive definite for z such that $R_2(z) > 0$. If $Q_2(z) > 0$ and $R_2(z) > 0$ then

$$P(z) = \begin{bmatrix} \eta_1(z) & \eta_2(z) \\ \eta_2(z) & \eta_3(z) \end{bmatrix}$$

is a common solution to A_1 and A_2.

The matrices $Q_2(z)$ and $R_2(z)$ are positive definite if its leading principle minors are positive. Define the functions

$$\begin{aligned} f_1(z) &= \gamma_1(z), \\ f_2(z) &= \gamma_1(z)\gamma_3(z) - \gamma_2^2(z), \\ f_3(z) &= z_1 z_3 - z_2^2, \\ f_4(z) &= z_1. \end{aligned} \qquad (4)$$

These functions are multivariate polynomials.

Let's give some basic properties of positive definite matrices. From $P = [p_{ij}] > 0$ it follows that $u^T P u > 0$ for all $0 \neq u \in \mathbb{R}^n$. Taking $u = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ we acquire

$$p_{ii} > 0 \quad (i = 1, 2, \ldots, n). \qquad (5)$$

The positive definite matrices comprise the cone interior. Therefore, the entry z_3 of $R_2(z)$ can be taken $z_3 = 1$.

Define the box $B = [0,1] \times [-1,1]$. If the matrices A_1 and A_2 have a common positive definite solution, there exists an $x \in B$ such that $P(x)$ is a common solution also.

In the case of three and higher dimensional matrices, there is no theoretical solution. To solve the problem, there are gradient based numerical algorithms (see [6,7]). In this work, we propose an algorithm which is based on sign-definite decomposition.

This paper differs from the mentioned works since it is determined a subbox which contains common solutions.

Let A_1 and A_2 be three-dimensional matrices. Assume that A_1 and A_2 have a common positive definite solution.

As in the two-dimensional case, common solution can be acquire.

For the matrices,

$$P(x) = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}, Q_3(y) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{bmatrix},$$

$$R_3(z) = \begin{bmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{bmatrix},$$

where $x, y, z \in \mathbb{R}^6$, the solution of the Lyapunov equations

$$\begin{aligned} A_1^T P(x) + P(x) A_1 &= -Q_3(y), \\ A_2^T P(x) + P(x) A_2 &= -R_3(z) \end{aligned} \tag{6}$$

is

$$x = (\phi_1(y), \phi_2(y), \dots, \phi_6(y))$$

and

$$x = (\eta_1(z), \eta_2(z), \dots, \eta_6(z)) \tag{7}$$

respectively.

As in the two-dimensional case, the matrix $Q_3(z)$ can be constructed

$$Q_3(z) = \begin{bmatrix} \gamma_1(z) & \gamma_2(z) & \gamma_3(z) \\ \gamma_2(z) & \gamma_4(z) & \gamma_5(z) \\ \gamma_3(z) & \gamma_5(z) & \gamma_6(z) \end{bmatrix}.$$

The functions that will be used to provide positive definiteness of $Q_3(z)$ and $R_3(z)$ are

$$\begin{aligned} g_1(z) &= \gamma_1(z), \\ g_2(z) &= \gamma_1(z)\gamma_4(z) - \gamma_2^2(z), \\ g_3(z) &= \det Q_3(z), \\ g_4(z) &= z_1 z_4 - z_2^2 \\ g_5(z) &= \det R_3(z) \\ g_6(z) &= z_1 \end{aligned} \tag{8}$$

So, these functions are multivariate polynomials.

4. AN APPLICATION OF SIGN-DEFINITE DECOMPOSITION

The sign of a multivariate polynomial function $h(a)$ over a box can be given by decomposition (see [17]).

Define the box

$$D = \{a \in \mathbb{R}^k : \ a_i^- \le a_i \le a_i^+, \quad i = 1,2,\dots,k\}.$$

Given any box can be transported to the first orthant in the parameter space. Therefore, one can assume that $a_i^- \ge 0$ without loss of generality. Then $h(a)$ can be written as

$$h(a) = h^+(a) - h^-(a)$$

where $h^+(a) \ge 0$ and $h^-(a) \ge 0$ for all $a \in D$. The functions $h^+(a)$ and $h^-(a)$ correspond to the positive and negative coefficients of $h(a)$.

Define the two extreme vertices of the box D

$$\begin{aligned} a^- &= (a_1^-, a_2^-, \dots, a_k^-), \\ a^+ &= (a_1^+, a_2^+, \dots, a_k^+). \end{aligned}$$

Lemma 3 ([17]): If $h^+(a^-) - h^-(a^+) > 0$ then $h(a) > 0$, If $h^+(a^+) - h^-(a^-) < 0$ then $h(a) < 0$ for all $a \in D$.

Using the sufficient conditions from Lemma 1, one can test positivity of f_i on the box B.

In order to apply this conditions, the B box (and accordingly the function f_i) must be transformed into the first orthant.

Here, we provide an algorithm for the common solution to 2-dimensional two matrices. We investigate the points $(z_1, z_2) \in B$ where the functions in (4) f_i $(i = 1,2,3,4)$ are positive. Here, we omitted the function $f_4(z)$.

This algorithm can also be adapted to 3-dimensional matrices. Notice that, among the functions g_i $(i = 1,2,\dots,6)$ given in (8), $g_6(z)$ can be omitted. In the case of existence, the following algorithm gives affirmative answer.

Algorithm 1:

Let B be initial box.

1. Calculate
   $m_i = f_i^+(b^-) - f_i^-(b^+)$,
   $M_i = f_i^+(b^+) - f_i^-(b^-) \quad (i = 1,2,3)$.
   where b^- and b^+ are extreme vertices of the box investigated.
2. If $\min\{m_1, m_2, m_3\} > 0$ then $f_1(z) > 0$, $f_2(z) > 0$ and $f_3(z) > 0$ for all z in the

box. Therefore, $Q_2(z) > 0$ and $R_2(z) > 0$. Otherwise, go to step 3.
3. If one of the M_i's is nonpositive, there is no z in the box so that $Q_2(z) > 0$ or $R_2(z) > 0$. This box should be eliminated. Otherwise, go to step 4.
4. The investigating box is divided into two subboxes along an axis. Return to step 1.

This processes are repeated until finding a subbox containing common solutions is provided or there is no subbox to be examined.

5. EXAMPLES

We will give the applications of the Algorithm 1.

Example 1. Consider the following Hurwiz stable matrices

$$A_1 = \begin{bmatrix} -2 & 4 \\ 1 & -7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix}.$$

The matrices $A_1 A_2$ and $A_1 A_2^{-1}$ have no negative real eigenvalues. From Theorem 3, they have common solution. The matrix $Q_2(z)$ is obtained with regard to z as

$$Q_2(z) = \begin{bmatrix} \frac{7}{9}z_1 - \frac{8}{9}z_2 + \frac{4}{9} & \frac{20}{9}z_2 - \frac{4}{9}z_1 - \frac{29}{18} \\ \frac{20}{9}z_2 - \frac{4}{9}z_1 - \frac{29}{18} & \frac{1}{3}z_1 + \frac{4}{3}z_2 + \frac{19}{3} \end{bmatrix}$$

and the common solution P can be written as

$$P(z) = \begin{bmatrix} \frac{2}{9}z_1 - \frac{1}{9}z_2 + \frac{1}{18} & \frac{1}{18}z_1 + \frac{2}{9}z_2 - \frac{1}{9} \\ \frac{1}{18}z_1 + \frac{2}{9}z_2 - \frac{1}{9} & \frac{1}{18}z_1 + \frac{2}{9}z_2 + \frac{7}{18} \end{bmatrix}$$

From the equation (4), we have the following functions

$$f_1(z_1, z_2) = \frac{7}{9}z_1 - \frac{8}{9}z_2 + \frac{4}{9},$$

$$f_2(z_1, z_2) = \frac{5}{81}z_1^2 + \frac{295}{81}z_1 - \frac{496}{81}z_2^2 + \frac{220}{81}z_1 z_2 + \frac{172}{81}z_2 + \frac{71}{324},$$

$$f_3(z_1, z_2) = z_1 - z_2^2.$$

The sign of the functions f_1, f_2, f_3 on the box $B = [0,1] \times [-1,1]$ is determined by Algorithm 1. After 116 steps in 0.022s, calculations give the following table. The calculations have been made by using the mathematical software Maple on a computer with an i5 1.4 GHz processor.

Table 1. Results of the elimination process

step	Subboxes	m	M_1, M_2, M_3	Operations
1	$[0,1] \times [-1,1]$	$-$	$+,+,+$	divide
2	$\left[0, \frac{1}{2}\right] \times [-1,1]$	$-$	$+,+,+$	divide
3	$\left[\frac{1}{2}, 1\right] \times [-1,1]$	$-$	$+,+,+$	divide
⋮	⋮	⋮	⋮	⋮
16	$\left[0, \frac{1}{4}\right] \times \left[-1, -\frac{1}{2}\right]$	$-$	$+,-,+$	eliminate
⋮	⋮	⋮	⋮	⋮
66	$\left[\frac{1}{4}, \frac{7}{8}\right] \times \left[-1, -\frac{1}{2}\right]$	$-$	$+,-,+$	eliminate
⋮	⋮	⋮	⋮	⋮
116	$\left[\frac{7}{8}, 1\right] \times \left[0, \frac{1}{4}\right]$	$+$	$+,+,+$	stop

For all $z \in \left[\frac{7}{8}, 1\right] \times \left[0, \frac{1}{4}\right]$, $P(z) > 0$ is common solution to A_1 and A_2. For instance, take $z = \left(\frac{15}{16}, \frac{1}{8}\right)$ the matrix

$$P = \begin{bmatrix} \frac{1}{4} & -\frac{1}{32} \\ -\frac{1}{32} & \frac{15}{32} \end{bmatrix}$$

is common solution to A_1 and A_2. When the Algorithm 1 continues for 2000 steps, the common solution set (subboxes) is obtained as in Figure 1.

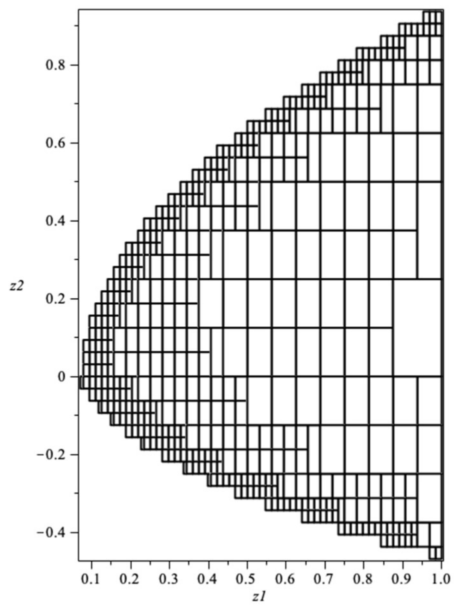

Figure 1. The subboxes are contained common solutions.

Example 2. Consider the following Hurwitz stable matrices

$$A_1 = \begin{bmatrix} -1 & -3 & -4 \\ 2 & -3 & -2 \\ 1 & 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -3 & 1 \\ 5 & 1 & -1 \\ -2 & 0 & -3 \end{bmatrix}.$$

Eigenvalues of $A_1A_2^{-1}$ and $L(A_1)L^{-1}(A_2)$, are

$$3.3244, \quad 0.3242 \pm 0.4138i,$$
$$0.5103, \quad 0.9922 \pm 1.0773i$$

respectively. We conclude that the segments $[A_1, A_2^{-1}]$ is stable by Theorem 2 since they have no negative real eigenvalues.

For initial box

$$B = [0,1] \times [-1,1] \times [-1,1] \times [0,1] \times [-1,1],$$

Algorithm 1 has given an affirmative result:

$$S = \left[\frac{17}{64}, \frac{35}{128}\right] \times \left[\frac{-1}{64}, \frac{-1}{128}\right] \times \left[\frac{-25}{64}, \frac{-3}{8}\right] \times \left[\frac{3}{128}, \frac{1}{32}\right] \times \left[\frac{1}{64}, \frac{1}{32}\right].$$

The matrix

$$P(z) = \begin{bmatrix} \eta_1(z) & \eta_2(z) & \eta_3(z) \\ \eta_2(z) & \eta_4(z) & \eta_5(z) \\ \eta_3(z) & \eta_5(z) & \eta_6(z) \end{bmatrix}$$

is the common solution of the Lyapunov equations (6) for all $z \in S$, where

$$\eta_1(z) = \frac{619}{3515}z_1 - \frac{113}{703}z_2 - \frac{258}{3515}z_3 + \frac{2659}{7030}z_4 - \frac{166}{3515}z_5 + \frac{86}{3515},$$

$$\eta_2(z) = \frac{273}{7030}z_1 - \frac{80}{703}z_2 - \frac{108}{3515}z_3 + \frac{1047}{3515}z_4 + \frac{94}{3515}z_5 + \frac{36}{3515},$$

$$\eta_3(z) = -\frac{18}{3515}z_1 + \frac{26}{703}z_2 + \frac{246}{3515}z_3 - \frac{83}{7030}z_4 + \frac{567}{3515}z_5 - \frac{82}{3515},$$

$$\eta_4(z) = \frac{819}{7030}z_1 - \frac{240}{703}z_2 - \frac{324}{3515}z_3 + \frac{2767}{7030}z_4 + \frac{282}{3515}z_5 + \frac{108}{3515},$$

$$\eta_5(z) = -\frac{219}{7030}z_1 + \frac{41}{703}z_2 - \frac{261}{3515}z_3 - \frac{106}{3515}z_4 + \frac{813}{3515}z_5 + \frac{87}{3515},$$

$$\eta_6(z) = \frac{61}{7030}z_1 - \frac{5}{703}z_2 + \frac{169}{3515}z_3 + \frac{43}{7030}z_4 - \frac{82}{3515}z_5 + \frac{1059}{7030}.$$

Here, $z_6 = 1$ can be taken (see equation (5)).

6. REFERENCES

[1] R. Bellman, Introduction to Matrix Analysis, SIAM, Philadelphia, 1997.

[2] H.K. Khalil, Nonlinear Systems, Prentice Hall, New Jersey, (2002).

[3] D. Liberzon, Switching in System and Control, Birkhauser, Boston, (2003).

[4] H. Lin, P.J. Antsaklis, Stability and stabilizability of switched linear systems: A survey of recent results, IEEE Transactions on Automatic Control, Vol.54, N.2, pp. 308-322, 2009.

[5] R.N. Shorten, K.S. Narendra, Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems, International Journal of Adaptive Control and Signal Processing, Vol.16, pp. 709-728, 2002.

[6] V. Dzhafarov, T. Büyükköroğlu, Ş. Yılmaz, On one application of convex optimization to stability of linear systems, Trudy Inst. Mat. i Mekh. UrO RAN, Vol.21, N.2, pp. 320-328, 2015.

[7] D. Liberzon, R. Tempo, Common Lyapunov functions and gradient algorithms, IEEE Transactions on Automatic Control, Vol.49, N.6, pp. 990-994, 2004.

[8] M. Góra, Some methods of construction of a common Lyapunov solution to a finite set of complex systems, Linear Algebra and its Applications, Vol.530, pp. 77-93, 2017.

[9] C. King, M. Nathanson, On the existence of a common quadratic Lyapunov function for a rank one difference, Linear Algebra and its Applications, Vol.419, pp. 400-416, 2006.

[10] T.J. Laffey, H. Šmigoc, Common solution to the Lyapunov equation for 2×2 complex matrices, Linear Algebra and its Applications, Vol.420, pp. 609-624, 2007.

[11] A.T. Fuller, Conditions for a matrix to have only characteristics roots with negative real parts, Journal of Mathematical Analysis and Applications, Vol.23, pp. 71-98, 1968.

[12] W. Govarets, B. Sijnave, Matrix manifolds and Jordan Structure of the bialternate matrix product, Linear Algebra and its Applications, Vol.292, pp. 245-266, 1999.

[13] L. Elsner, V. Monov, The bialternate matrix product revisited, Linear Algebra and its Applications, Vol.434, pp. 1058-1066, 2011.

[14] R.K. Yedevalli, Robust Control of Uncertain Dynamic Systems: A linear state space approach, Springer, New York, 2014.

[15] R.K. Yedevalli, Robust Control of Uncertain Dynamic Systems: A linear state space approach, Springer, New York, 2014.

[16] B.R. Barmish, New Tools for Robustness of Linear Systems, Macmillan, New York, 1994.

[17] L.H. Keel and S.P. Bhattacharyya, Robust stability via sign-definite decomposition, IEEE Transactions on Automatic Control, Vol.56, N.1, pp. 140-145, 2011.